Geometry PoW Packet

Spring Up, Fall Back

Problem 4707 • http://mathforum.org/pows/

Welcome!

This packet contains a copy of the problem, the "answer check," our solutions, some teaching suggestions, and samples of the student work we received in May 2008. The text of the problem is included below. A print-friendly version is available using the "Print" link on the problem page.

Check out the Problems of the Week blog, visit us on Facebook, join the *geopow-teachers* discussion and/or follow us on Twitter. You can always find the latest scoop at http://mathforum.org/community/.

Standards

In ***Spring Up, Fall Back***, students use trigonometry to determine the missing sides of a triangle formed by some solar panels mounted on a house.

If your state has adopted the Common Core State Standards, these alignments might be helpful.

High School: Geometry: Similarity, Right Triangles, and Trigonometry

8. Use trigonometric ratios and the Pythagorean theorem to solve right triangles in applied problems.
11. Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles.

High School: Geometry: Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Additional alignment information can be found through the Write Math with the Math Forum service, where teachers can browse by NCTM and individual state standards to find related problems.

The Problem

Spring Up, Fall Back

The solar panels mounted on the side of our porch can be set to two different angles. The top small picture shows the angle for winter, when the sun is lower in the sky. The bottom one shows the angle for summer, when the sun is higher.

Notice that there are two holes in the rails where they attach to the bottom of the panels, one hole for each angle. The rails and panels are mounted to the porch in such a way that they can swing up and down. The edge of the panel as shown in the small pictures is 18.5 inches long.

We'd like to make a new rail so that we can set the panels to 26, 44, or 61 degrees. That measurement refers to the angle the panel makes with a horizontal line from the porch, not to the angle the panel makes with the rail.

1. How far from the mounting point of the panel should we mount the new rail on the porch so that the rail and the porch form a 90 degree angle when the panel forms a 44 degree angle with the horizontal?
2. How far along the rail should we drill holes so that the panels can be positioned at 26, 44, or 61 degrees?

(Want to learn more about why we would want to change the angle? Check out this info from OkSolar.com: http://www.oksolar.com/technical/angle_orientation.html.)

Answer Check

After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually **get** the answer) we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

> The mounting point of the rail should be 12.85 inches from the mounting point of the panel. The first hole in the rail should be at about 9.57 inches.

If your answer **does not** match our answer,

- did you start with a right triangle with a hypotenuse of 18.5 inches and one angle of 44 degrees?
- did you try using some right triangle trigonometry?
- did you notice that for the 26 and 61 degree angles, you don't have a right triangle? You'll need to draw in a horizontal line yourself.

If any of these questions help you, you might **revise** your answer, and then leave a **comment** that tells us what you did. If you're still stuck, leave a **comment** that tells us where you think you need help.

If your answer **does** match ours,

- did you explain your work as well as you can?
- did you make any mistake? How did you find them?
- are there any hints that you would give another student?

Revise your work if you have any ideas to add. Otherwise leave us a **comment** that tells us how you think you did—you might answer one or more of the questions above.

Our Solutions

Method 1: Mathematical Model

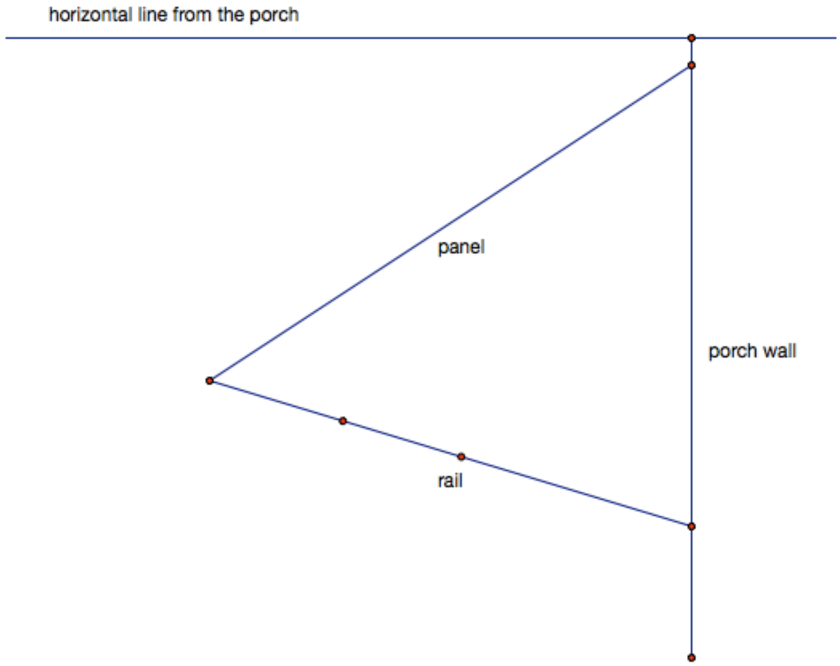
Plan:

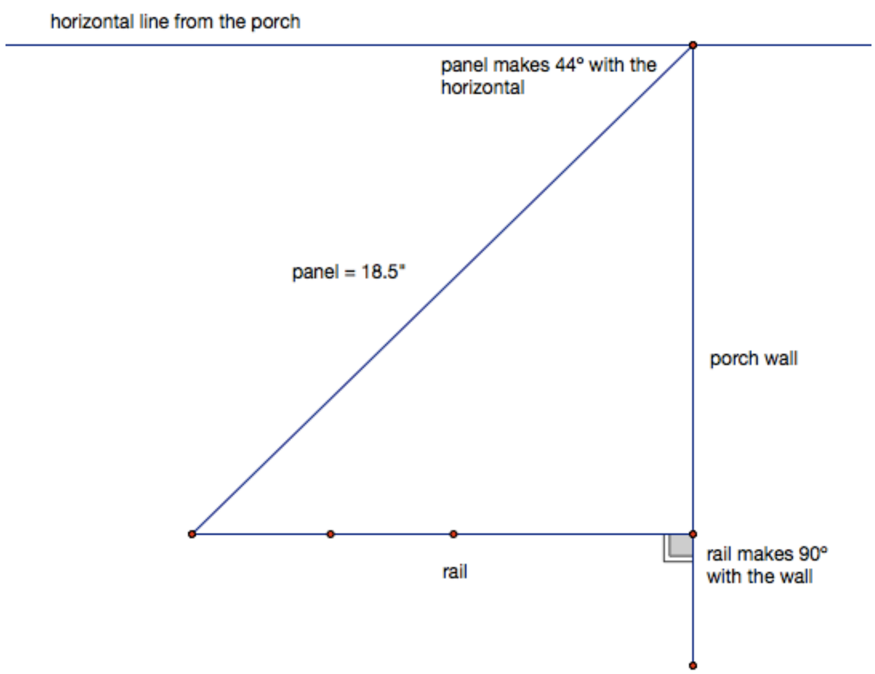
My first strategy will be to Understand the Problem. I will start with a sketch because I can't visualize all of the words and clues in the problem. I'm wondering if my sketch should be to scale and accurate with angles or not. I'm worried I could sketch the problem wrong and get a wrong answer because I don't understand the story.

After that I might need to make an actual to scale drawing to solve the problem, using the Change the Representation strategy. Or I might need to Make a Mathematical Model to decide what calculations to do with the angles and lengths. I will probably need to use trigonometry since it's dealing with angles and lengths.

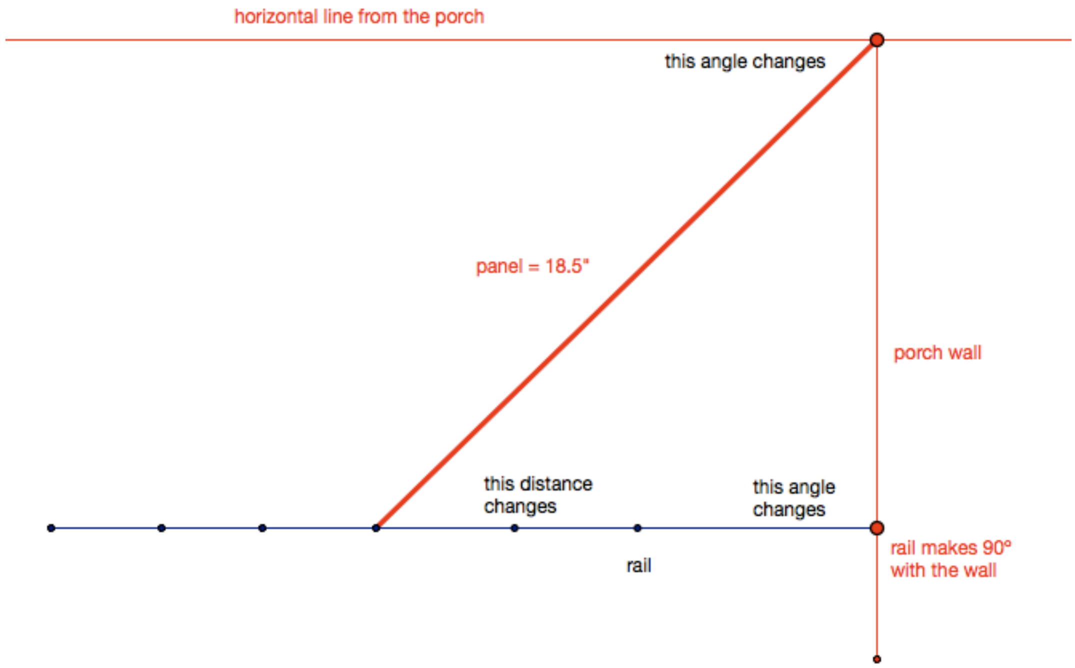
Carrying out:

My sketches





When I changed the angles, lots of things changed. What can move and what can't move in the sketch?



Checking in on the plan:
-I think my drawing matches the picture, because I could change the movable parts to match the photos, without messing up the parts that aren't supposed to move.
-At first I thought the top of the panel could slide up and down, but from the photos I realized it can't
-I'm glad I thought to check the photos!

Where do we go from here:
-Thinking about what can change and can't change made me think of variables and constants in a mathematical model
-I know how to find the values for the right triangle situation because I can use trigonometry with one angle and one side to find a missing angle.
-I can use a scale drawing or maybe some more trig to solve part 2.

Mathematical Model:
The hypotenuse of the right triangle is 18.5", and the angle opposite the wall is 44° because it's an alternate interior angle with the 44° angle formed by the wall and the horizontal line (the rail is parallel to the horizontal line when it's perpendicular to the wall).

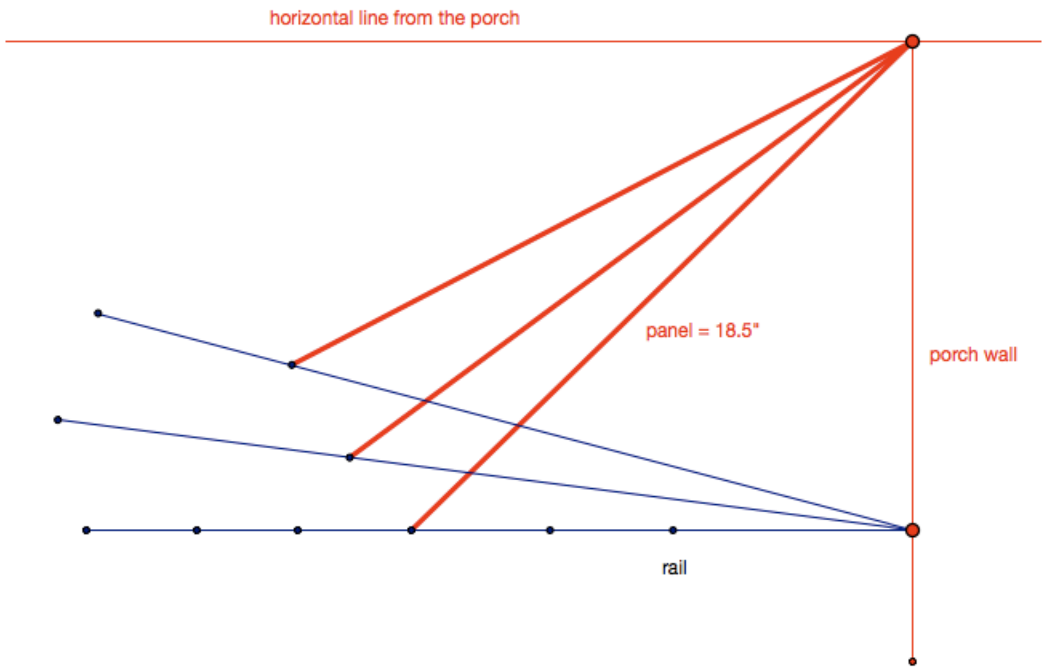
I'm trying to find the side adjacent to the 44° angle so I can use cos(44°).

Model:

$$\cos 44° = \frac{adj}{hyp} = \frac{x}{18.5}$$
$$x = 18.5\cos 44°$$
$$x = 13.3"$$

So I can make a hole in the rail 13.3 inches along its length and when the rail is perpendicular to the wall, the panel will be at an angle of 44° with the horizontal.

Now how do I solve part 2? I will start with more sketches. Maybe one can turn into a scale drawing.

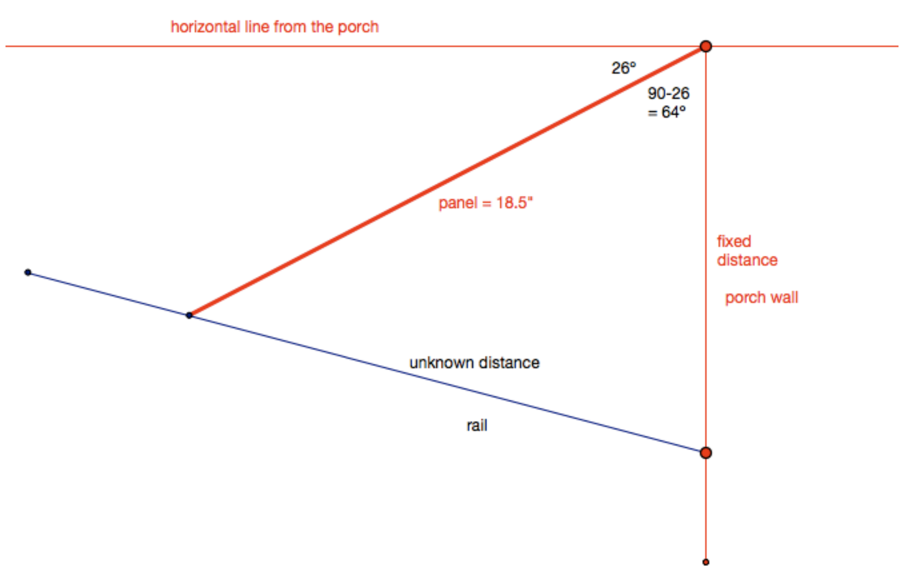


I can tell the rail will have to get longer to make the angle smaller. But I no longer have right triangles, so I can't use regular trig. Maybe I could use the law of sines or the law of cosines, which deal with non-right triangles.

$$\textit{Law of Sines}: \frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$
$$\textit{Law of Cos}: a^2 + b^2 - 2ab\cos(C) = c^2$$

What measures do I know?

I know the length of one side is 18.5"
I know the size of one angle is... wait a minute, the angle I'm given isn't even in these triangles. Yikes! I need to look at more pictures. I should make another picture with known and unknown quantities labeled like I did for part 1.



Okay, I found out some other quantities with fixed values that I can calculate, and I found out the known and unknown quantities match what I need for Law of Cosines.

Plan: Use the previous problem to find the distance from where the rail is mounted to where the panel is mounted. Then use the Law of Cosines to find the length of the panel when the angle with the horizontal is 44º and 61º.

Step 1: Finding the distance between where the rail and the panel are mounted.

The distance is part of a right triangle with hypotenuse of 18.5" (the panel) and 13.3" (the distance that I found when the rail makes a right triangle) as the other leg.

Model:

$$\begin{aligned} 18.5^2 &= 13.3^2 + x^2 \\ 18.5^2 - 13.3^2 &= x^2 \\ 165.36 &= x^2 \\ 12.9 &= x \end{aligned}$$

Okay, got that. What was that for again? Oh yeah, using the Law of Cosines.

Step 2: Finding where to drill the hole on the rail for the Law of Cosines.

Wait a sec, just to double check -- am I finding the right thing? Did the author want to know how long to make the rail, or did she already have a rail and wanted to know where to attach it to the wall? I better reread the problem.

Dang, I solved the wrong problem at first! But I guess it doesn't really matter since once the angle and hypotenuse were chosen, there was only one length for the rail and only one "mounting distance" that would work. I guess I just did extra work... So the rail should be mounted 12.9" below where the panels are mounted, and the Law of Cosines will tell me what lengths to drill the other two holes at:

$$\textit{Law of Cos}: a^2 + b^2 - 2ab\cos(C) = c^2$$

$$\begin{aligned} a &= 12.9" \\ b &= 18.5" \\ c &= x \\ \cos(\textit{angle}\, C) &= \cos(64°) \end{aligned}$$

$$\begin{aligned} c^2 &= 12.9^2 + 18.5^2 - 2(12.9)(18.5)\cos(64°) = 299.4 \\ c &= 17.3" \end{aligned}$$

I think it makes sense that side would be longer... Let me try for when angle c is 90-61 = 29º.

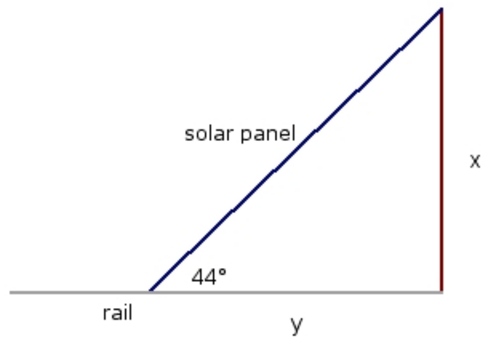
$$\begin{aligned} a &= 12.9" \\ b &= 18.5" \\ c &= x \\ \cos(\textit{angle}\, C) &= \cos(29°) \end{aligned}$$

$$\begin{aligned} c^2 &= 12.9^2 + 18.5^2 - 2(12.9)(18.5)\cos(29°) = 91.2 \\ c &= 9.6" \end{aligned}$$

Okay, I think I have solved the problem:

The rail should be mounted 12.9" below where the tops of the panels are mounted, and should have holes drilled at 17.3", 13.3", and 9.6" along its length.

Method 2: Make a Different Mathematical Model



Question 1: The panel, porch, and rail form a right triangle with the angle between the rail and the panel being 44 degrees.

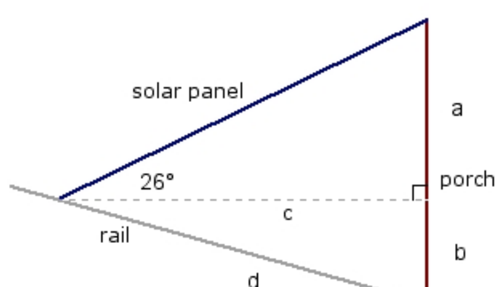
We can use the sine ratio to find the length of the vertical of the triangle, since that is opposite the known angle, and the other known side is the hypotenuse.

$$\sin = \frac{opp}{hyp}$$
$$\sin 44 = \frac{x}{18.5}$$
$$12.85 = x$$

The rail should be mounted 12.85" below the mounting point of the panel.

Question 2: For the 44 degree situation, we can use the picture above. But now, since we want the side that's adjacent to the 44 degree angle, we can use cosine.

$$\cos = \frac{adj}{hyp}$$
$$\cos 44 = \frac{y}{18.5}$$
$$13.31 = y$$



The hole in the rail for the 44 degree triangle should be 13.31 inches from the porch end.

For the 26 degree situation, we need to use a couple of different right triangles, as shown in the picture below.

We can first find a using the sine of 26.

$$\sin 26 = \frac{a}{18.5}$$
$$8.11 = a$$

We can find b by subtracting a from the length of the whole side, which we found to be 12.85 inches.

$$b = 12.85 - 8.11$$
$$b = 4.74$$

We can find c using the cosine of 26.

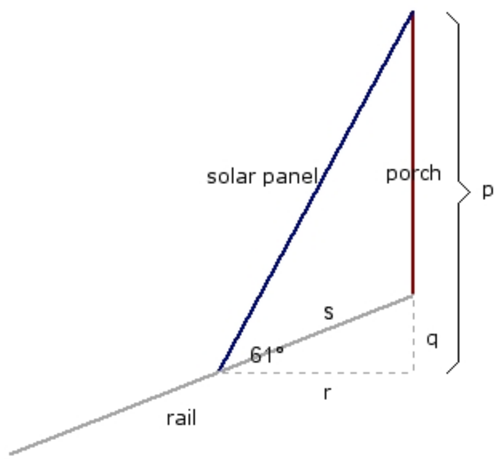
$$\cos 26 = \frac{c}{18.5}$$
$$16.63 = c$$

Now we can find d using the Pythagorean theorem.

$$d^2 = 4.74^2 + 16.63^2$$
$$d = 17.29$$

The hole in the rail for the 26 degree situation should be 17.29 inches from the porch end.

For the 61 degree situation, we can again use a couple of different right triangles, as shown in the picture below.



First, find p using the sine of 61.

$$\sin 61 = \frac{p}{18.5}$$
$$16.81 = a$$

Next, find q by subtracting 12.85 from p.

$$q = 16.18 - 12.85$$
$$q = 3.33$$

Find r using the cosine of 61.

$$\cos 61 = \frac{r}{18.5}$$
$$8.97 = r$$

Finally, find s using the Pythagorean theorem.

$$s^2 = 3.33^2 + 8.97^2$$
$$s = 9.57$$

The hole in the rail for the 61 degree situation should be 9.57 inches from the porch end.

Teaching Suggestions

This problem came about because it was time for us to change the angle on our solar panels, and I had long known that there was some math involved! As shown in the problem, our panels are set up to be adjusted twice a year (at the spring and fall equinox). I did some research and found that while changing the angle twice a year can increase the efficiency of the panels, changing more often might gain even more efficiency. I figured students could do the math of the actual construction. (We've since gotten bigger ones, and while we still need to change the angle seasonally, it's not as simple as drilling some holes in angle iron!)

It's not necessary to know *why* anyone should do this, but of course you could use that as a related science lesson. The link given in the problem is one place you can read more about it. To see how we figured out the required angles, you can find a number of angle calculators on the web. You could even have students calculate appropriate angles for your location (the latitude of our house is about 44° North).

It's possible that many students aren't that familiar with right triangle trig, and even fewer will know the Law of Cosines. Looking at **Our Solutions**, it seems like the solution using only right triangle trigonometry is more complex, and requires the addition of auxiliary lines. That's a great strategy that students could practice. But if they're saying, "But we don't have a right triangle, and it would be really messy to make right triangles…isn't there an easier way?", that seems like a moment that calls for the Law of Cosines! You might even send them off to do some research on their own about solving non-right triangles.

The most important part of this problem, however, is illustrated nicely in the first solution. Just what exactly is going on, anyway?? The idea of the panels pivoting at the top where they attach to the deck, and the various angles and lengths changing as they do that, is crucial to devising a strategy to find a solution. Students might try drawing some pictures or, maybe better, modeling with some physical objects that maintain their lengths as angles change so that students don't get duped by a drawing in which they've changed lengths or angles that shouldn't have changed.

In **Our Solutions**, we've used our Make a Mathematical Model strategy. Check it out for more hints about how to help students get started. You'll find everything linked from the Problem Solving Activities link in the left menu bar when you're logged in.

The Online Resources Page for this problem contains links to related problems in the Problem Library and to other web-based resources.

If you would like a calendar of the Current Problems, consider bookmarking this page:

http://mathforum.org/pow/support/

Sample Student Solutions

focus on **Completeness**

In the solutions below, I've provided the scores the students would have received in the **Completeness** category of our scoring rubric, which is shown below. My comments focus on what I feel is the area in which they need the most improvement.

Novice	Apprentice	Practitioner	Expert
Has written very little that explains how the answer was achieved.	Submitted explanation without work or work without explanation. Leaves out enough details that a fellow student couldn't follow or learn from the explanation.	Explains most of the steps taken to solve the problem, and the rationale for them, with enough detail for another student to understand. This might include • which angles were used in a given triangle • what lengths were used where and why • how any angles or lengths that weren't given were calculated	Adds in useful extensions and further explanation of some of the ideas involved. The additions are helpful, not just "I'll say more to get more credit"

Student88
age 11

Completeness
Novice

90 inches from the mounting point. 26 inches, 44 inches and 61 inches.

i did something

Student88 has used the given angles as lengths, and it's not clear whether they actually did *do something as they claim. I might ask them if they could draw a picture of the situation in Question 1 and offer to help them with that if they aren't sure how to do that.*

Cara
age 15

Completeness
Novice

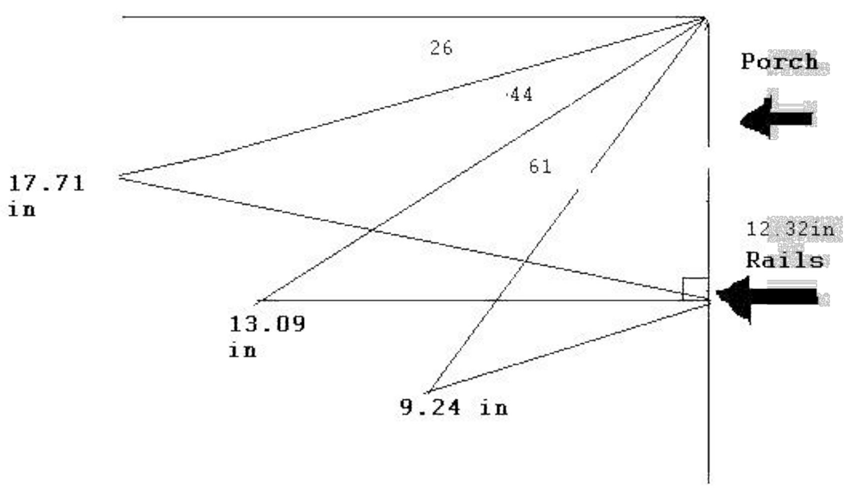
You should mount the rail 12.32 inches from the mounting point. For 26 degrees drill a hole at 17.71 in. For 44 degrees drill a hole at 13.09 in. For 61 degrees drill a hole at 9.4 in.

The solar panels mounted on the side of our porch can be set to two different angles. The top small picture shows the angle for winter, when the sun is lower in the sky. The bottom one shows the angle for summer, when the sun is higher.

Notice that there are two holes in the rails where they attach to the bottom of the panels, one hole for each angle. The rails and panels are mounted to the porch in such a way that they can swing up and down. The edge of the panel as shown in the small pictures is 18.5 inches long.

We'd like to make a new rail so that we can set the panels to 26, 44, or 61 degrees. That measurement refers to the angle the panel makes with a horizontal line from the porch, not to the angle the panel makes with the rail.

How far from the mounting point of the panel should we mount the new rail on the porch so that the rail and the porch form a 90 degree angle when the panel forms a 44 degree angle with the horizontal? How far along the rail should we drill holes so that the panels can be positioned at 26, 44, or 61 degrees?



Cara has included the text of the problem and a drawing of her understanding of the situation, but has written not a single work that explains how she found her answers. I might ask her to explain how she decided that the rail should be mounted 12.32 inches from the mounting point and go from there.

Student893
age 13

Completeness
Apprentice

1. You have to mount the new rail 12.85 inches from the mounting point of the panel. 2. You have to drill holes 16.63 inches, 13.31 inches, and 8.97 inches from the porch.

1. The angle from the panel to the porch is 46 degrees.

cos46 = h / 18.5
h = cos 46 * 18.5
h = 12.85

2.
sin64 = L1 / 18.5
L1 = sin64 * 18.5
L1 = 16.63

Student893 definitely has some good ideas. I'm now sure if the 46 degree angle is a typing error or a calculation mistake, since s/he doesn't explain the origin of the equations. I might ask them to tell me more about how they found h – what do all those numbers represent?

sin46 = L2 / 18.5
L2 = sin46 * 18.5
L2 = 13.31

sin29 = L3 / 18.5
L3 = sin29 * 18.5
L3 = 8.97

You have to drill holes 16.63 inches, 13.31 inches, and 8.97 inches from the porch.

Gordon
age 16

Completeness
Apprentice

The hole in the rail from the mounting point to achieve a 44 degree angle with the horizontal is 13.3 inches.

Since the rail is to make a 90 degree angle with the porch. then it will be the horizontal line I can measure the angle for my solar cell. So I have a right triangle with an acute angle of 44 degrees and I need to know the length adjacent to the panel which is 18.5 inches long. So I used the equation cos(44) = x/18.5, multiplying by 18.5 gives me x = 18.5 cos(44) = approx. 13.3 inches. I then used pythagorean theorem to get the distance between the panel and its hinge with the hinge for the rail and got sqrt(18.l5^2 - 13.3^2) = approx. 12.85 inches.

I then used the law of cosines to solve the other two angels in part two. For 26 the hole is 17.3" away from the hinge on the porch and for 61 the hole is 9.6 inches from the hinge.

Gordon has written a decent explanation of his answer to the first question. But I'd like to hear more about how he found the other answers at the end, so I'd simply ask him to include more detail and see how that goes.

Mona
age 15

Completeness
Practitioner

The holes are at 13.31 inches, 17.29 inches, and 9.57 inches.

To answer question 1, I made a right triangle because the problem says that the rail and the porch should form a right angle. The angle between the rail and the panel is 44 degrees. I know the panel is 18.5 inches long. To find the side of the triangle that is the porch, I can use sine, because I want the opposite and I have the hypotenuse.

sin44=opposite over hypotenuse

sin44=x/18.5

x = 12.85

The rail should be mounted 12.85 inches below where the panel is mounted.

To answer question 2, I can use the same triangle. This time I used cosine to find the bottom.

cos44=x/18.5

x=13.31

If you drill a hole in the rail that's 13.31 inches from the porch, you'll get the 44 degree angle.

At first I thought the 26 degree one would work the same way, but then I saw that it isn't a right triangle. I drew a horizontal line from the bottom of the solar panel and it made two right triangles. The top one has a 26 degree angle and a hypotenuse of 18.5 inches. I want to find the hypotenuse of the bottom right triangle.

First I found the bottom of the top triangle using cosine.

cos26 = x/18.5

I might encourage Mona to learn about the Law of Cosines and see if she can see how she could use it to solve this problem.

x = 16.63 inches

Then I found the other side of the top triangle using sine.

sin26 = x/18.5

x = 8.11

The right side of the bottom triangle is 12.85 minus 8.11. That's 4.74 inches.

I used the Pythagorean Theorem to find the third side.

c^2 = 4.74^2 + 16.63^2

c = 17.29

To make the 26 degree angle, drill another hole 17.29 inches from the porch. The picture I used for the third one is like adding a little right triangle to the picture of the solar panel, the rail, and the porch. The big right triangle has an angle of 61 degrees and a hypotenuse of 18.5 inches. I can find the right side using sine.

sin61=x/18.5

x=16.18

I can find the bottom using cosine.

cos61=x/18.5

x=8.97

I can find the right side of the smaller right triangle, since it is the side of the large right triangle minus the distance on the porch.

16.18-12.85=3.33

I used the Pythagorean Theorem to find the hypotenuse of the small right triangle.

x^2 = 3.33^2 + 8.97^2

x = 9.57

The last hole should be at 9.57 inches. They probably don't need to be that accurate. Like they could do 13.25 inches, 17.25 inches, and 9.5 inches. That is easier on a ruler anyway and would still be really close.

Mohammad
age 17

Completeness
Expert

We should mount the new rail almost 12.9" far from the mounting point of the panel on the porch so that the rail and the porch form a 90 degree angle when the panel forms a 44 degree angle with the horizontal.

We are speaking about solar panel ,porch and rails which are actually three dimensional objects ,but when we look at them from one side we see solar panel and rails as lines as well as the side of porch .So along given angles we can draw two dimensional lines as figures 1 , 2 and 3 .

1. Figure 1 depicts what have been given by question 1 .We are going to find measure of AB .What is added to the shape except data is measure of angle A1 .Since porch is implicitly perpendicular to earth surface the measure of angle A1 is 90 degrees .

Two line which are perpendicular to a line are parallel . When a line cross two parallel line make equal angles with them .Hence measure of angle C is equal to 44 degrees as well .

We are where it is apparently necessary to trigonometry .We know measure of AC, length of edge of the panel . What is left to do is calculating sin44 .However let's find an relatively accurate answer without trigonometry .Using sin44 can be left as checking the solution later .

Mohammad's explanation doesn't leave much to the imagination! I like his point about looking at this as a two-dimensional figure. He is also very throrough with the details. I don't have very many suggestions to make to him.

Since 44 degrees is very close to 45 degrees ,we can replace it by 45 degrees . By this assumption triangle ABC is isosceles then:

AB = BC

Using Pythagorean theorem we will get to:

AB^2 + BC^2 = AC^2

2*AB^2 = AC^2

AC = 18.5 "

2*AB^2 = 18.5^2

AB = 18.5 / (2^ 0.5)

AB # 13.1 "

We should mount the new rail almost 13" far from the mounting point of the panel on the porch so that the rail and the porch form a 90 degree angle when the panel forms a 44 degree angle with the horizontal.

2. This question is about distance between holes on the rail and porch ,BC .

For the case 44 degrees we saw the triangle ABC is close to an isosceles one and AB = BC so :

We should drill holes almost 13 " far along the rail so that the panels can be positioned at 44 degrees .

For the case 61 degrees ,figure 3 , we can again suppose 60 instead of 61 degrees .In this way there is no point to use trigonometry . measure of angle A3 # 90 -60 =30 degrees

It is proved that in a right angle triangle if one of angles has measure 30 degrees ,the length of its front side is half of the length of hypotenuse .

BC # (1/2)* AC

BC # (1/2)* 18.5 = 9.3 "

We should drill holes almost 9.25 " far along the rail so that the panels can be positioned at 61 degrees .

Figure 2 depicts the left case , 26 degrees .Apparently there is not a near measure to use geometry theorems .Let's figure out that by trigonometry.

cos 26 = BC / AC # 0.9

BC / 18.5 # 0.9

BC # 16.6 "

We should drill holes almost 16.6 " far along the rail so that the panels can be positioned at 26 degrees .

Checking : As mentioned earlier ,trigonometric ratios can lead us to more accurate results . In figure 1 :

sin 44 = AB /AC # 0.69

AB # 12.9"

cos 44 = BC / AC # 0.72

BC # 13.3"

In figure 2 :

cos 26 = BC / AC # 0.90

BC # 16.6"

As done previously .

In figure 3 :

cos 61 = BC / AC # 0.48

BC # 9.0 "

The modified values are as following .

We should mount the new rail almost 12.9" far from the mounting point of the panel on the porch so that the rail and the porch form a 90 degree angle when the panel forms a 44 degree angle with the horizontal.

We should drill holes almost 13.3 " far along the rail so that the panels can be positioned at 44 degrees .

We should drill holes almost 16.6 " far along the rail so that the panels can be positioned at 26 degrees .

We should drill holes almost 9.0 " far along the rail so that the panels can be positioned at 61 degrees .

Scoring Rubric

A **problem-specific rubric** can be found linked from the problem to help in assessing student solutions. We consider each category separately when evaluating the students' work, thereby providing more focused information regarding the strengths and weaknesses in the work. A **generic student-friendly rubric** can be downloaded from the Teaching with PoWs link in the left menu (when you are logged in). We encourage you to share it with your students to help them understand our criteria for good problem solving and communication.

We hope these packets are useful in helping you make the most of Geometry Problems of the Week. Please let me know if you have ideas for making them more useful.

~ Annie

