Ladder Problems Revisited

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Introduction

The so-called ladder problems have circulated in the recreational mathematical literature for a long time. There are two main versions of these problems (to be defined below): 1) single ladder problems (also called the box problem) and 2) crossed ladders problems.

The straightforward solution approach leads in these problems to a quartic equation. Since a quartic equation can be solved exactly by algebra, these problems are in a sense trivial. The main reason for the great interest in these problems in recreational mathematics is the challenge to avoid to solve a quartic equation, using the exact solution methods which involves cumbersome algebra with cube root extractions.

Recently, an analysis of these problems has been published ([1] - [3]). In addition to give a comprehensive discussion (including a literature review) of the ladder problems, these papers seem to contain several original ideas; and to this author's best knowledge, are the first attempt to systematically discuss and analyze the ladder problems 1) and 2) in detail.

Based on the discussion in [1], [2] and [3] we formulate the exercises below. Whenever an exercise is referenced below, it is neither given in [1], [2] nor [3].

Single Ladder Problem

The single ladder problem (SLP) -also known as the box problem- may be formulated as follows [4]:

A ladder of length c is leaning against a wall in such a manner that one point of the ladder is just touching a box which has an $a \times b$ cross section and is pushed against the wall. How much (x) of the ladder is between the wall and the point of the contact?[1]

In the following, let $a > 0$, $b > 0$, $c > 0$, $d = \sqrt{a^2 + b^2}$, $k = \sqrt{a^2 + c^2}$ ($d > 0, k > 0$) and $t = \sqrt[3]{b/a}$ real. If $a = b$ we have the *square* box problem. We denote the problem above with $SLP[a, b, c]$.[2]

The straightforward solution approach for the box problem leads to a quartic equation for x, the so-called corresponding quartic equation: $x^4 - 2cx^3 + (c^2 - a^2 - b^2)x^2 + 2a^2cx - a^2c^2 = 0$. Then x is a solution of the box problem $SLP[a, b, c]$ if and only if x is a root of this quartic equation and

[1] In [4] only the case $a = b$ is discussed.

[2] The single ladder problem is well known in the recreational mathematical literature. The oldest reference to the square box problem we know is from 1745. (Thomas Simpson: *A Treatise of Algebra*, John Nourse, London 1745, page 250.) For the general box problem the oldest reference we know is from 1907. (A. Cyril Pearson: *The Twentieth Century Standard Puzzle Book*, George Routledge & Sons, Ltd., London 1907, page 103.)

$a < x < c - b$.

The (Legendre-)resolvent of the corresponding quartic equation is called the resolvent of the box problem.

Exercise 1 Show that the elimination of the cubic term of the corresponding quartic equation for the box problem gives an equation quadratic in form if and only if the cross section of the box is a square (i.e. $a = b$).

Exercise 2 By exercise 1 the square box problem is exact solvable using square roots only. Show, using exercise 1, that the two different solutions are $\frac{1}{2}(c \pm \sqrt{(k-3a)(k-a)}\,)$ when $c > 2\sqrt{2}\,a$, that the one and only one solution is $\sqrt{2}\,a$ when $c = 2\sqrt{2}\,a$, and that there is no solution when $c < 2\sqrt{2}\,a$.

Exercise 3 For the box problem, we define y as $xy = ac$ and get a quartic equation for y. What is the geometrical meaning of y? For the square box problem, show that the quartic equation for y may be factorized as $[y^2 - (a+k)y + a(a+k)][y^2 - (a-k)y + a(a-k)] = 0$. (This factorization was obtained in [1], using the resolvent of the quartic equation. The same approach has been used earlier in [9]). Show that the elimination of the cubic term of the quartic equation for y never gives an equation quadratic in form, as in exercise 1.

Exercise 4 For one of the solutions of the square box problem, we may let $0 < \theta < \pi/2$ be the angle between the ladder and the wall. Show that $\sin 2\theta$ is given by a quadratic equation.

Exercise 5 Let $xy = ac$ as in exercise 3 and $y = a(u+1)$ for the box problem. Show that for the square box problem, $u + 1/u$ is given by a quadratic equation (i.e. the quartic equation for u is reciprocal or palindromic). What is the geometrical meaning of u? Show that in general for the box problem $SLP[a,b,c]$, the quartic equation for u is only reciprocal for the square box problem, i.e. when $a = b$.

Exercise 6 Let z be the height of the ladder above the ground for the box problem. Show that for the square box problem, $y + z$ (for y as given in exercise 3) is given by a quadratic equation. (This solution of the square box problem is very old (it belongs to this problem's folklore). It is given in *Simpson* as early as 1745 (see footnote 2 on page 1).)

Exercise 7 A quartic equation may be solved using the Ferrari method. (Many standard algebra books give details on this method - or see [1].) Using this method, the roots of the corresponding quartic equation for the box problem $SLP[a,b,c]$ can be written as $x_{1,2} = \frac{c}{2} + \frac{R}{2} \pm \frac{D}{2}$ and $x_{3,4} = \frac{c}{2} - \frac{R}{2} \pm \frac{E}{2}$ with $R = \sqrt{\tau + d^2}$ and for τ a root of the resolvent of the box problem. (The details of how R, D and E can be find are given in [5].) Show that for the corresponding quartic equation for the box problem, we may choose $R = 0$ if and only if we have the square box problem.

Exercise 8 Use the results of the exercises 3 - 7 to find the solution of the square box problem as given in exercise 2. For the exercise 7, use both $R = 0$ and $R \neq 0$.

Exercise 9 Show that for a and b fixed there exists a c_0, $a + b < c_0 < (a+b)\sqrt{2} \leq 2d$ ($c_0 = 2d$ if and only if $a = b$), such that the box problem $SLP[a,b,c]$ has one and only one solution when $c = c_0$, two different solutions when $c > c_0$, and no solution when $c < c_0$.

Exercise 10 Show that the c_0 from exercise 9 is $c_0 = a\ (1+t^2)^{\frac{3}{2}}$ and that the solution then is given as $x_0 = a\,\sqrt{1+t^2}$.

Exercise 11 Show that box problem $SLP[n, n+1, m]$ for $n > 0$ and $m > 0$ integers has two solutions if $m > 3n+2$ and no solutions if $m < 2n+1$. What happens when $2n+1 \leq m \leq 3n+2$?

Exercise 12 For $0 < \theta < \pi/2$, solve the box problem $SLP[\cos^3\theta, \sin^3\theta, 1]$ exactly.

Exercise 13 Let c be fixed and consider the box problems $SLP[a, b, c]$, all having one and only one solution. Relate these problems to the *astroid*, see [6]. Explain that the box problem $SLP[a, b, c]$ has one and only one solution if and only if $c^{2/3} = a^{2/3} + b^{2/3}$, to relate the general box problem to the astroid. Show that all such box problems are given as $SLP[c\cos^3\theta, c\sin^3\theta, c]$ for $0 < \theta < \pi/2$. (See also exercise 12.)

Exercise 14 Show that to express the solution of the box problem $SLP[a, b, c]$ exactly, complex cube roots (or trigonometric functions) must be used: In general, show that it is not possible to express the solution of the box problem exactly using only real root extractions.

Exercise 15 The box problem $SLP[a, b, 2d]$ for $a \neq b$ has one trivial solution. Use that to show that the non-trivial solution is given as $(1 - 2\cos(\theta + \pi/3))d$ where $\cos 3\theta = (b^2 - a^2)/d^2$. Also, show that if a, b and d are integers, then θ cannot be rational (as given in degrees).

In general the box problem cannot be solved geometrically, using a straight-edge and compasses only.[3] In [3], geometrical solutions of the box problem are discussed in depth.

Exercise 16 The square box problem is exact solvable using square roots only, so the problem - when it has a solution - is geometrically solvable. Show how, using both exercise 3 and exercise 6, to solve the square box problem geometrically, when a and c are lengths of given line segments. Also, discuss the number of solutions of the square box problem geometrically.

Exercise 17 Find examples of box problems $SLP[a, b, c]$ for a, b ($a \neq b$) and c integers and with to different solutions such that 1) both solutions and 2) none of the solutions can be constructed, using only a straight-edge and compasses.

Exercise 18 Assume that the box problem SLP[a,b,c] has at least one solution. Show that the SLP then is geometrically solvable if and only if the resolvent of the SLP has at least one root which can be constructed.[4]

Exercise 19 Let a and b be integers, and assume that the corresponding box problem $SLP[a, b, c]$ has one and only one solution. Then show that the box problem can be solved geometrically, using only a straight-edge and compasses if and only if both a/d and b/d are cubes, where $d = gcd(a, b)$.

Exercise 20 Explain that when the box problem in exercise 19 is geometrically solvable and a and b are relative prime, then the (real) cube root of $a + b$ must always be irrational. Finally, give explicitly infinite many examples of such box problems, where all $a + b$ are square numbers.

Exercise 21 Let the box problem $SLP[a, b, c]$ be as in exercise 19 and geometrically solvable. Show that with a and b be relative prime and c an integer, a, b and c all are cubes. Also, show that then the solution of the box problem and y and z (as given in exercise 3 and 6, respectively) all

[3] We say that the box problem is geometrically solvable if and only if *both* solutions can be constructed, using a straight-edge and compasses only.

[4] We assume that a, b and c are lengths of line segments which can be contructed, using only a straight-edge and compasses.

are integers. Finally, show that the resolvent of the problem must have three integer roots. Also, show that there is one positive double root and one negative single root of the resolvent.

Crossed Ladders Problem

The crossed ladders problem (CLP) involves two ladders and may be formulated as follows [7]:

Two ladders, of length a and b cross at a point c above a lane. Each ladder reaches from the base of the wall to some point on the opposite wall. How wide (x) is the lane?

We denote the CLP above with $CLP[a, b, c]$.[5] We assume that $a > b > c > 0$, $K = a^2 - b^2$ and that the longest ladder reaches the opposite wall at the height y above the lane.

The straightforward solution approach for the CLP leads to a quartic equation for y, the so-called corresponding quartic equation for the CLP: $y^4 - 2cy^3 - Ky^2 + 2cKy - Kc^2 = 0$. Then $x = \sqrt{a^2 - y^2}$ is a solution of the CLP if and only if y is a root of this quartic equation and $2c < y < a$.

The (Legendre-)resolvent of the corresponding quartic equation is called the resolvent of the CLP.

Exercise 22 Show that the elimination the cubic term of the corresponding quartic equation for the $CLP[a, b, c]$ gives an equation quadratic in form if and only if a, b and c are related as the sides of a right triangle.

Exercise 23 By exercise 22 the CLP for the case $a^2 = b^2 + c^2$ is exact solvable using square roots only when it has a solution. Show that the solution is then given as $x = \sqrt{a^2 - y^2}$ for $y = \frac{c}{2}(1 + \sqrt{5 + 4\sqrt{2}})$ (as long as $y < a$).

Exercise 24 For given a and b, show that the $CLP[a, b, c]$ has at most one solution. For $c < \frac{ab}{a+b}$, show that there is exactly one solution; otherwise there is no solution. (This was discussed in [1], but has been already given in [8].)

Exercise 25 Show -without solving the CLP explicitly- that if the $CLP[a, b, c]$ has a solution, then to express this solution exactly, only real root extractions (square roots and *one* cube root) are needed. See also exercise 26 below.

Exercise 26 Show that the exact solution of the $CLP[a, b, c]$ -when it exists- may be written as

$$x = \sqrt{a^2 - y^2},\ y = \tfrac{1}{2}\left(v + \tfrac{K}{v}\right) \text{ and } v = \sqrt{\tfrac{K}{\sqrt{3}}}\left[\sqrt{\sqrt{4t^2 + 3} - t} - \sqrt{t}\right]$$

where $t = \sinh\frac{\theta}{3}$ for $\sinh\theta = \frac{3\sqrt{3}}{K}\,c^2$.

Exercise 27 Show that for $K = c^2$, exercise 26 gives the solution of the CLP as given in exercise 23.

Exercise 28 Let the shortest ladder of the CLP reach the opposite wall at the height z above the lane. Show that when $CLP[a, b, c]$ has a solution z satisfies $2c - K/(8c) < z < 2c - Kc/(8c^2 + K)$. (This problem is from [8].)

[5] The crossed ladders problem is well known in the recreational mathematical literature. The oldest reference to the CLP we know is from 1894. (Problem 24. *The American Mathematical Monthly*, 1 (1894), 353–354.)

Exercise 29 Let a, b and c all be integers such that $CLP[a,b,c]$ has a solution. Determine all such cases of the CLP where also x, y and z (z is defined as in exercise 28) are integers. (The problem is solved in [10].) Also, find infinite many such examples of the CLP where in addition the distance between the top of the ladders are integers. (The problem is solved in [11]. See also the discussion in [12] and [13].) Does it exist a box problem $SLP[a,b,c]$ with a, b ($a \neq b$), c integers and two different solutions, where both solutions also are integers?

Exercise 30 Solve the problem $CLP[a,b,2\sqrt{K/3}]$ exactly.

In general, the CLP cannot be solved geometrically, using a straight-edge and compasses only.[6] In [3], geometrical solutions of the crossed ladders problem are discussed in depth.

Exercise 31 Show that, by giving explicitly infinite examples $CLP^n = CLP[a_n, b_n, c_n]$ such that: i) all of the CLP^n has a solution, ii) a_n, b_n and c_n for all of the CLP^n are line segments which can be constructed, and iii) none of the CLP^n can be solved geometrically. (HINT: CLP^n may be chosen as $CLP[a_n, b_n, c_n]$ for $c_n = 2\sqrt{K_n/3}$ (see exercise 30) where $K_n = {a_n}^2 - {b_n}^2$ and a_n, b_n and c_n are defined as follows for $n > 1$: $r = 2n+1$, $s = n(r+1)$, $t = s+1$, $a_n = \sqrt{3}\,t$, $b_n = \sqrt{3}\,s$ and $c_n = 2r$.)

Exercise 32 Show that the problem $CLP[a,b,c]$ can be solved geometrically, when it has a solution -using a marked ruler only- when a, b and c all can be such constructed, by giving the actual construction. (This problem was discussed in [3] but is originally from [14].) What can be said about this for the non-square box problem $SLP[a,b,c]$?

Exercise 33 Is a similar result to exercise 18 for the SLP also true for the CLP? Explain!

Ladder Problems and Mathematica™

The exercises below illustrate the use of Mathematica [15] for the solution of the ladder problems SLP and CLP.[7]

Exercise 34 Explain that the following Mathematica commands solve the square box problem

```
Q[x_,a_,b_,c_] :=
x^4 - 2 c x^3 + (c^2 - a^2 - b^2) x^2 + 2 a^2 c x - a^2 c^2

x /.  Solve[Q[x,a,a,c] == 0,x] [[{1,2}]] // Simplify
```

Exercise 35 Execute the Mathematica commands above, and compare the result with solution of the SLP given in exercise 2.

Exercise 36 Try the Mathematica commands above with $a \neq b$, i.e. examine the general SLP using Mathematica. Comments?

Exercise 37 Explain that the following Mathematica commands

```
q[y_,a_,c_] :=
```

[6] We say that the CLP is geometrically solvable when it has a solution if and only if its only solution can be constructed, using a straight-edge and compasses only. (We assume that a, b and c can be constructed using a straight-edge and compasses only.)

[7] None of the exercises below are found in [1] - [3], which do not discuss the use of Mathematica.

```
y^4 - 2 a y^3 + (2 a^2 - c^2) y^2 + 2 a c^2 y - a^2 c^2
PolynomialReduce[q[y,a,c],k^2 - a^2 - c^2,{c}] // Factor // Last;
TraditionalForm[Map[Collect[#,y,Factor]&,%]]
```

give a factorization of q as the product of two quadratic polynomials. (See exercise 3.)

Exercise 38 Execute the Mathematica commands above.

Exercise 39 Using q as above, explain why the Mathematica command:

```
Factor[q[y,a,c],GaussianIntegers -> True]
```

does not give the factorisation in exercise 37, but that

```
Factor[q[y,1,10],GaussianIntegers -> True]
```

does for the special case $a = b = 1$, $c = 10$ of the square box problem $SLP[1, 1, 10]$.

Exercise 40 What happens in the last command above if the option GaussianIntegers is not used?

Let r, s and t be positive real numbers such that $r^2 + s^2 = t^2$.

Exercise 41 Examine the box problem $SLP[s, 3r, 2t\ ]$ using Mathematica. Show that when the problem has two solutions, the solutions are roots of the cubic equation
$x^3 - 5tx^2 + 8s^2x - 4s^2t^2 = 0$.

Exercise 42 Examine the box problem $SLP[rs, rs, (r + s)t\ ]$ using Mathematica. Also, solve the problem exactly using both geometry and algebra.[8]

Exercise 43 Examine the box problem $SLP[r^2, s^2, (r + s)t\ ]$ using Mathematica. Also, solve the problem exactly using both geometry and algebra.

Exercise 44 Examine the box problem $SLP[r^3, s^3, t^3\ ]$ using Mathematica. Also, solve the problem exactly using both geometry and algebra. Compare with the ladder problem in exercise 12.

Exercise 45 Examine the box problem $SLP[r, (k - 1)s, kt\ ]$ (here $k > 1$ and real; not as defined on page 1) using Mathematica. Also, solve the problem exactly using both geometry and algebra. Show that when r, s, t and k are integers, the corresponding box problem always has two solutions. How are the problems in this exercise and exercise 44 related?

Exercise 46 Examine the ladder problem $CLP[rt^3, st^3, r^2s^2\ ]$ using Mathematica. Also, solve the problem exactly using both geometry and algebra. (This problem is from [16].)

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