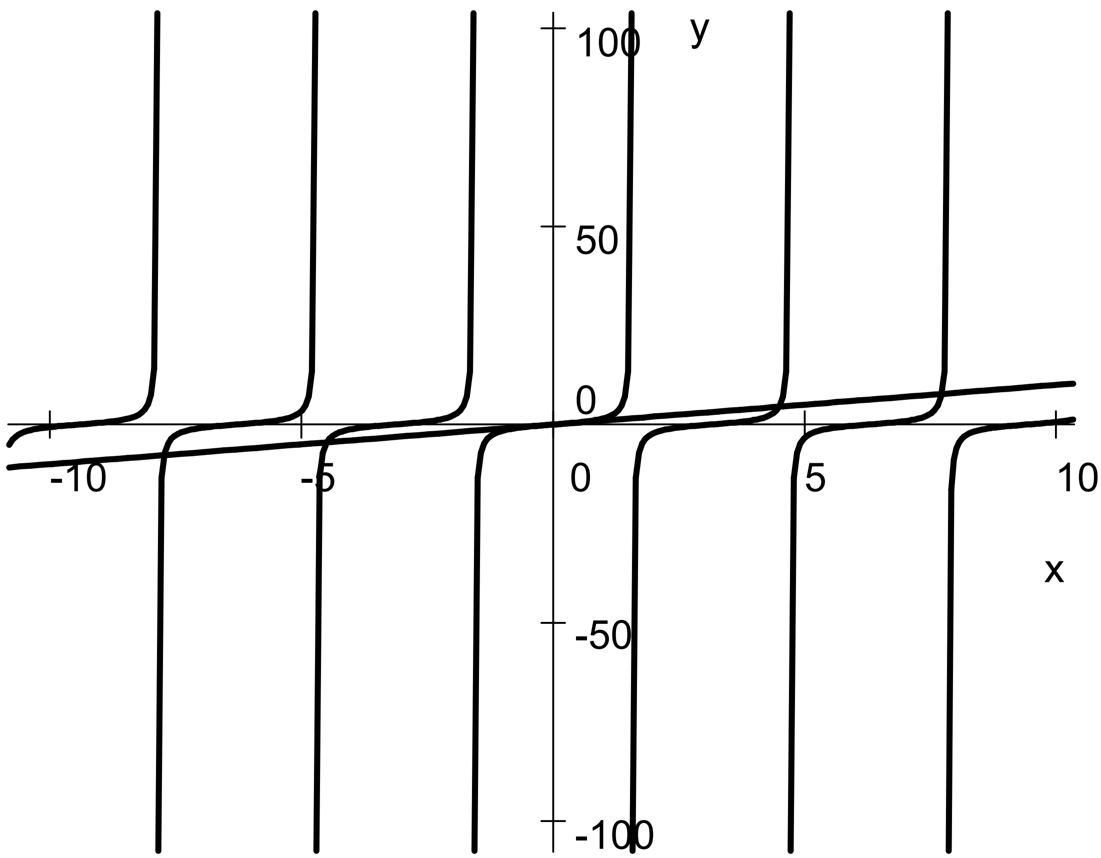
The Remarkable Equation $\tan x = x$



$y = \tan x$ and $y = x$

Some Numerical Values

n	λ_n	$\frac{1}{2}(2n+1)\pi$
1	4.4934094579	4.7123889804
2	7.7252518369	7.8539816340
3	10.9041216594	10.9955742876
4	14.0661939128	14.1371669412
5	17.2207552719	17.2787595947
6	20.3713029593	20.4203522483
7	23.5194524987	23.5619449019
8	26.6660542588	26.7035375555
9	29.8115987909	29.8451302091
10	32.9563890398	32.9867228627

$$\lambda_n < \frac{1}{2}(2n+1)\pi \qquad (n = 1,\ 2,\ 3,\ \dots)$$

$$\frac{1}{2}(2n+1)\pi - \lambda_n \longrightarrow 0$$

Fixed Points by Iteration

- Each λ_n is a repelling fixed point of $\tan x$. (derivative is greater than 1 near each λ_n)
- However, iteration works for $\arctan x = x$. (derivative is less than 1 near each λ_n)
- Within $\left(\frac{\pi(2n-1)}{2}, \ \frac{\pi(2n+1)}{2}\right)$, the iterates of $\arctan x + n\pi$ converge to λ_n.

Table of k^{th} iterates when $b = 4.5$

k	$\tan x$	$\arctan x + \pi$
2	2.28220445019	4.49342411319
4	−2.29654896062	4.49340949055
6	2.10347056093	4.49340945798
8	7.92538965777	4.49340945791
10	−6.31908514080	4.49340945791

Calculus Appearances

- The turning points of $y = \frac{1}{x}\sin x$ satisfy $\tan x = x$.

- The turning points of $y = x^a \sin x$ $(a \neq 0)$ satisfy $\tan x = -\frac{1}{a}x$.

- The turning points of $y = x \sin \frac{1}{x}$ are at $x = \pm\frac{1}{\lambda_1}, \pm\frac{1}{\lambda_2}, \pm\frac{1}{\lambda_3}, ...$

- The (rapidly damping) turning points of $e^{-\frac{1}{2}ax^2} \sin x$ and $e^{-\frac{1}{2}ax^2} \cos x$ are given by the nonzero solutions to $\cot x = ax$ and $\tan x = -ax$, respectively.

More Calculus Appearances

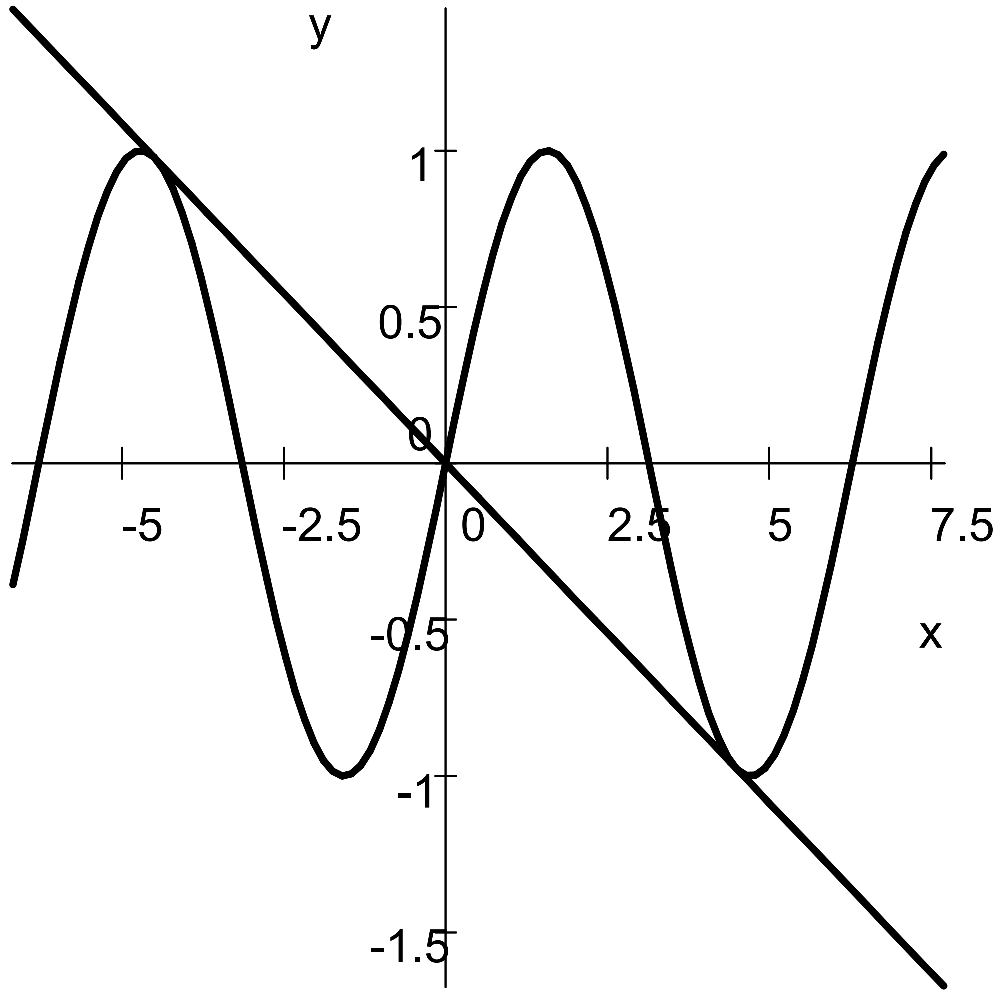
- Illustrating the **Fundamental Theorem of Calculus**:

$$n\pi \; + \int_0^{\lambda_n} \frac{1}{1+x^2}\,dx \quad = \quad \lambda_n$$

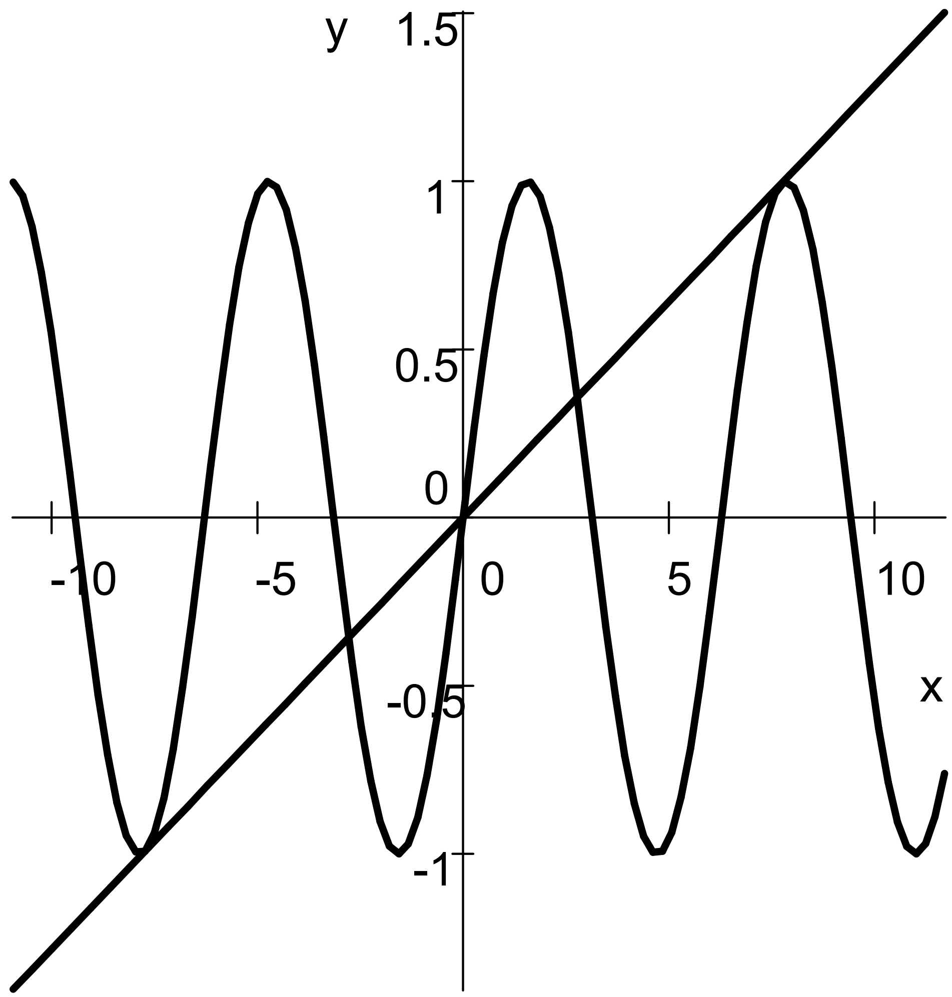
We have to add $n\pi$ is because λ_n belongs to the n^{th} complete branch of the tangent function for $x > 0$.

- The solutions to $\tan x = x$ give the points where the tangent line (TL) to $y = \sin x$ passes through the origin.

- The x and y intercepts of the involute of $x^2 + y^2 = r^2$ that starts at $(r, 0)$ are found by solving $\tan\theta = \theta$ and $\cot\theta = -\theta$, respectively.



$y = \sin x$ and TL at $x = \pm\lambda_1$



$y = \sin x$ and TL at $x = \pm\lambda_2$

Physics and Applied Math Appearances

- Conduction of heat in a sphere.
- Bound state energies in quantum mechanics for a particle in a finite square well potential.
- The equation $\tan(ax+b) = x$ arises in the molecular field theory of ferromagnetism.
- The solutions to $\tan x = x$ give a nice illustration of the general behavior of eigenvalues in a Sturm–Liouville problem.
- The zeros of the Bessel functions $J_{n+\frac{1}{2}}(x)$ are solutions to $\tan x = R_n(x)$, where $R_n(x)$ is a rational function with integer coefficients.

  The zeros of $J_{\frac{3}{2}}(x)$ are solutions to $\tan x = x$.

All Solutions to $\tan x = x$ are Real

- **Theorem 1** *If* $a \in \mathbb{R}$, *then* $\tan z = az$ *has no nonreal solutions if* $a \geq 1$ *or* $a \leq 0$, *and exactly two nonreal solutions (both pure imaginary) if* $0 < a < 1$.

Proof (Hardy, p. 480): Let $z = x + iy$ and equate real and imaginary parts.

$$\frac{\sin 2x}{\cos 2x + \cosh 2y} = ax$$

$$\frac{\sinh 2y}{\cos 2x + \cosh 2y} = ay$$

If $xy \neq 0$, we have the contradiction

$$1 > \frac{\sin 2x}{2x} = \frac{\sinh 2y}{2y} > 1$$

Thus, $y = 0$, which gives only real roots, or $x = 0$, which gives $\tanh y = ay$, from which the nature of the nonreal roots follows (consider graphically).

- Proof by Rouché's theorem: Greenleaf (pp. 414–416) and Hille (pp. 255–256).
- Proof by theorems about eigenvalues for certain Sturm–Liouville problems: Carslaw/Jaeger (pp. 324–326) and Ziegler.

[1] Horatio Scott Carslaw and J. C. Jaeger, **Conduction of Heat in Solids**, 1959.

[2] Frederick P. Greenleaf, **Introduction to Complex Variables**, 1972.

[3] Godfrey H. Hardy, **A Course of Pure Mathematics**, 1952/1996.

[4] Einar Hille, **Analytic Function Theory**, Volume I, 1959.

[5] Martin Ziegler, *Solution II to Monthly Problem #E1857,* American Mathematical Monthly **74** #6 (June/July 1973), 723.

Each nonzero Solution to $\tan x = x$ is Transcendental

Proof: Lindemann's theorem (special case): *If α is nonzero and algebraic, then e^α is transcendental*.

$$e^{2ix} = \frac{e^{ix}}{e^{-ix}} = \frac{\cos x + i \sin x}{\cos x - i \sin x} = \frac{1 + i \tan x}{1 - i \tan x}$$

If $\lambda \neq 0$ and $\tan \lambda = \lambda$, then

$$e^{2i\lambda} = \frac{1 + i\lambda}{1 - i\lambda}.$$

If λ were algebraic, then $2i\lambda$ is nonzero and algebraic and $\frac{1+i\lambda}{1-i\lambda}$ is not transcendental, which contradicts Lindemann's theorem.

[1] Milton Brockett Porter, *On the roots of the hypergeometric and Bessel's functions*, American Journal of Mathematics **20** (1898), 193–214.

More General Transcendental Results

The same idea works for $\tan x = R(x)$, where $R(x)$ is a non–identically zero rational function with rational coefficients. Also, when both occurrences of "rational" are replaced with "algebraic". The latter implies, for example, that the nonzero solutions to $\sin x = ax$, for all algebraic a such that $0 < a < 1$, are transcendental. This last equation arises in problems of finding the corresponding circle radius and arc angle for a given arc length and its corresponding chord length, and in problems involving how deep a log of a certain specific gravity and a specific diameter sinks in water.

Stronger Transcendentality Notions

Chow describes a result due to F.–C. Lin which says that if Schanuel's Conjecture is true, then these numbers do not belong to the algebraic closure of the collection of all numbers that can be explicitly expressed using rational numbers and the elementary functions (and their inverses) of calculus.

[1] Timothy Y. Chow, *What is a closed–form number?*, American Mathematical Monthly **106** #5 (May 1999), 440–448.

Asymptotic Expansion for λ_n

The following was independently obtained by

- Euler (1748, pp. 318–320; pp. 323–324 of French translation)
- Cauchy (1827, p. 272; pp. 277–278 in **Oeuvres**, Series 1, Volume 1)
- Rayleigh (1877, p. 334)

$$\lambda_n \sim \alpha_n - \alpha_n^{-1} - \tfrac{2}{3}\alpha_n^{-3} - \tfrac{13}{15}\alpha_n^{-5} - \tfrac{146}{105}\alpha_n^{-7} - \cdots$$

$$\alpha_n = \tfrac{1}{2}(2n+1)\pi$$

[1] Augustin–Louis Cauchy, **Théorie de la Propagation des Ondes à la Surface d'un Fluide Pesant d'une Profondeur Indéfinie**, 1827.

[2] Leonhard Euler, **Introductio in Analysin Infinitorum**, Volume 2, 1748.

[3] John William Strutt Rayleigh, **The Theory of Sound**, Volume 1, Macmillan, 1877.

Curious Sums Involving λ_n

Recall that $\sum_{n=1}^{\infty} n^{-2}$ converges to $\frac{1}{6}\pi^2$.

Rayleigh (read June 11, 1874) proved

$$\sum_{n=1}^{\infty} (\lambda_n)^{-2} = \tfrac{1}{10}$$

$$\sum_{n=1}^{\infty} (\lambda_n)^{-4} = \frac{1}{350} \qquad \sum_{n=1}^{\infty} (\lambda_n)^{-6} = \frac{1}{7875}$$

$$\sum_{n=1}^{\infty} (\lambda_n)^{-8} = \frac{37}{6,063,750} \qquad \sum_{n=1}^{\infty} (\lambda_n)^{-10} = \frac{59}{197,071,875}$$

[1] John William Strutt Rayleigh, *Note on the numerical calculation of the roots of fluctuating functions*, Proceedings of the London Mathematical Society (1) **5** (1873–74), 119–123.

Information for my Records

Iowa Section of The Mathematical Association of America
April 7–8 (Friday/Saturday), 2006
Iowa State University (Ames, Iowa)
10:15–10:35 A.M. April 8

Abstract Submitted

Although $\tan x = x$ is virtually the prototypical example for solving an equation by graphical methods, and this equation frequently appears in calculus texts as an example of Newton's method, there seems to be nothing in the literature that surveys what is known about its solutions. In this talk I will look at some appearances of this equation in elementary calculus, some appearances of this

equation in more advanced areas (quantum mechanics, heat conduction, etc.), the fact that this equation has no nonreal solutions and that all of its nonzero solutions are transcendental, and some curious infinite sums involving its solutions. In addition, I will discuss some of the history behind this equation, including contributions by Euler (1748), Fourier (1807), Cauchy (1827), and Rayleigh (1874, 1877).