

## Reflections on the History of Complex Numbers

The first occurrence of square roots of negative quantities in Western literature can be found in *Ars Magna* by Cardan (1545) in the solution of the classical problem: Find two numbers whose sum and product are given. In the case the sum is 10 and the product 40, Cardan acknowledges that there is no solution but proposes to extend the calculation in the following way: share 10 into two quantities,  $5 + m$  and  $5 - m$ , where  $m$  is a mysterious quantity so that  $(5 + m)(5 - m) = 25 - m^2 = 40$ ; therefore  $m^2 = -15$ . The emergence of square roots of negative numbers also occurred in examining solutions of equations of the form  $x^3 = px + q$ . The Tartaglia-Cardan formula for solving such equations cannot be applied in the irreducible case when  $4p^3 - 27q^2 < 0$ . This case leads to the square root of a negative quantity. Bombelli, in *Algebra* (1572) makes a decisive step when in solving the equation:  $x^3 = 15x + 4$ , he introduces “something” whose square is  $-1$ , which he names: “*piu di meno*” and formulates rules for calculating with it.

From that time, this “thing” was used in calculations, first in intermediate steps for finding real number solutions, then for expressing impossible solutions allowing the generalization of results and theorems in the theory of equations. For instance, it allowed Albert Girard in 1629, to give the first formulation of the Fundamental Theorem of Algebra in the *Invention nouvelle en l’algèbre*: “Toutes les equations d’algèbre reçoivent autant de solutions que la denomination de la plus haute quantité le démontre” (in modern terms, all polynomial equations have as many roots as the degree of the polynomial). In 1637, Descartes in the *Geometry*, introduces the expression “*imaginary quantities*” which was used until Gauss (1831) introduced the terminology “*complex numbers*” and clarified the status of these quantities. In much of the 18<sup>th</sup> century, complex numbers were used in calculations but their meaning was the object of metaphysical discussions as was the case for infinitesimals. Many contradictions arose from applying the usual rules of calculations with square roots when using the notation  $\sqrt{-a}$  with  $a$  positive, Euler introduced the notation “*i*” in the *Elements d’algèbre* (1774), but this notation, which was also used by Gauss, was not automatically accepted.

The question of the meaning of imaginary quantities was solved through their geometric interpretation at the beginning of the 19<sup>th</sup> century. Geometric interpretations were independently proposed by different, though not necessarily well-known mathematicians: Argand, l’Abbé Buée in France, Wessel in Denmark, and then Gauss. In a letter to Bessel in 1811, Gauss proposed to use the plane for representing all quantities, both reals and imaginaries, by attaching to the point of coordinates  $(a, b)$  the quantity  $a + bi$ . The construction by Argand (1806) was different as he started from the determination of proportional means between two quantities ( $x$  such that  $a:x::x:b$ , or as  $a/x = x/b$ ). In the case where  $a$  and  $b$  have different signs, for instance  $a = 1$  and  $b = -1$ , he pointed out the

impossibility of solving the problem while staying on the real line, and proposed to think in terms of directions in the plane as in Figure 1.

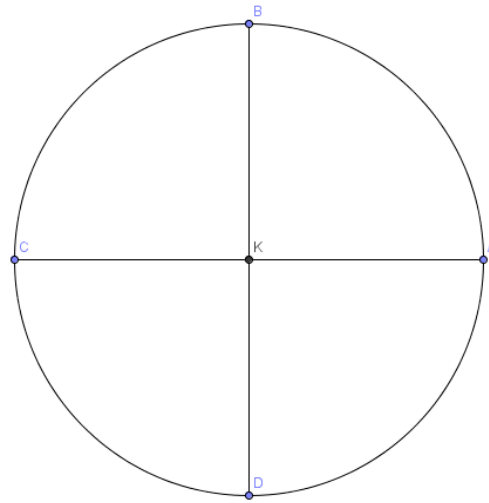


Figure 1

Thus, if the directed segments  $\overrightarrow{KB}$  and  $\overrightarrow{KC}$  are associated with the numbers 1 and  $-1$ , the directed segments  $\overrightarrow{KB}$  and  $\overrightarrow{KD}$  appear as the solutions of the equation  $1:x::x:-1$ , or  $\frac{1}{x} = \frac{x}{-1}$ . Argand pointed out that these solutions are usually denoted by  $\sqrt{-1}$  and  $-\sqrt{-1}$ .

Then Argand defined operations between directed segments: addition through the parallelogram law, which corresponds to the addition of vectors associated with the directed segments, and multiplication as in Figure 2 for two unitary directed segments  $\overrightarrow{KB}$  and  $\overrightarrow{KC}$  as  $\overrightarrow{KD} = \overrightarrow{KB} \cdot \overrightarrow{KC}$ ,  $\angle AKB = \angle CKD$ , and thus  $KA:KB::KC:KD$ .

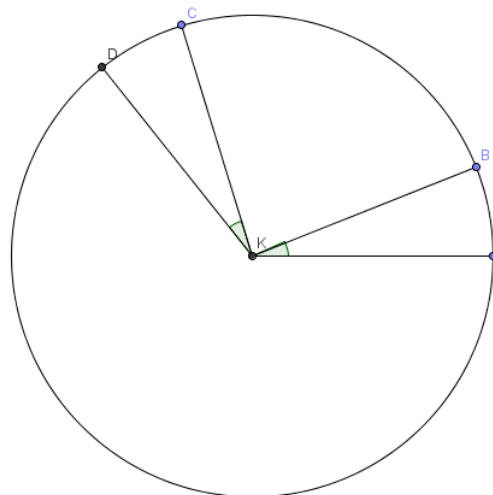


Figure 2

For non-unitary directed segments, the lengths are multiplied. He thus developed a calculation on directed segments that provided a geometrical interpretation of imaginary quantities and of the calculations with these. In this construction, angles and the link between multiplication and rotation played a fundamental role.

The 19<sup>th</sup> century saw new constructions of complex numbers, this time of an algebraic nature. This was the case with Cauchy, whose ambition in his *Mémoire sur la théorie des équivalences algébriques substituées à la théorie des imaginaires* (1847) was to “se débarrasser complètement des expressions imaginaires, en réduisant la lettre  $i$  à n’être plus qu’une quantité réelle [to completely banish imaginary expressions by reducing the letter  $i$  to be only a real quantity].” For doing so, he referred to the theory of congruences by Gauss and considered the quotient of the set of real polynomials  $\mathbb{R}[X]$  by the polynomial  $X^2 + 1$ . Each class in this quotient is a representative of the form  $aX + b$  corresponding to the rest in its division by  $X^2 + 1$ . Cauchy extended the operations on  $\mathbb{R}[X]$  to this quotient, showing that the sum of the class of  $a + bX$  and  $c + dX$  is the class of  $(a + c) + (b + d)X$ , and that the product of these classes is  $(ac - bd) + (ad + bc)X$  as the polynomial  $X^2$  is congruent to the polynomial  $-1$ . Thus in Cauchy’s theory, “ $i$ ” becomes a real but undetermined quantity.

Another algebraic construction was proposed by Hamilton in the *Theory of Conjugate Functions and Algebraic Couples* in 1833. Complex numbers were defined as couples of real numbers  $(a, b)$ , where each real number was associated with the couple of the form  $(a, 0)$ ; addition was defined as the ordinary addition of couples and multiplication such that there exists a couple whose square is  $-1$ . This led to the following definition for the product  $(a, b) \bullet (c, d) = (ac - bd, ad + bc)$ . In this system the number “ $i$ ” that Hamilton named the secondary unit is the couple  $(0,1)$ . After many unsuccessful attempts to extend this construction to the three-dimensional space through the use of triplets, Hamilton eventually created the field of quaternions, a non-commutative field where geometrical notions such as scalar product and vector product find a natural interpretation.

This brief describes only a small part of the rich history of complex numbers and does not touch the theory of functions of complex variables also initiated by Cauchy, and the way this theory contributed to a number of mathematical domains including number theory, differential geometry, and dynamical systems. This brief history illustrates how readers might approach and discuss many important issues from a mathematical and epistemological perspective including why and how new mathematical objects are introduced, the diverse ways they can become legitimate and meaningful, the potential and limitation of symbolic choices, the role of representations and connections between these, and the importance of connections between domains. The many investigations and connections that can be considered using complex numbers suggests the study of complex numbers should have a place in teacher education. Furthermore, well structured and cognitively rather simple ideas encapsulated in complex numbers form a persuasive

argument to keep the rich intuitive, geometric and algebraic meanings of complex numbers as a part of high school mathematics teaching syllabi.

### **Reference**

Commission Inter-IREM Epistémologie et Histoire des Mathématiques. *Images, Imaginaires, Imaginations - Une perspective historique pour l'introduction des nombres complexes*. Paris: Editions Ellipses, 1998.