

# A Learning Progression for Complex Numbers

In mathematics curriculum development around the world, the opportunity for students to study complex numbers in secondary schools is decreasing. Given that the study of complex numbers offers a space for the development of such mathematical ideas as relationships between algebra and geometry and the extension of the real number system, this brief offers suggestions in a variety of contexts that curriculum developers might use to incorporate the study of complex numbers in secondary school mathematics.

A curriculum developer is an individual who defines standards and expectations at a broader perspective, i.e., individuals who work in policy. At the same time a curriculum developer is an individual engaged in developing textbooks, and a curriculum developer is a classroom teacher choosing mathematical topics to teach secondary students. A central element in developing curriculum is the need to think about how the ideas relating to a particular concept unfold over time. This brief describes a possible progression of the concepts related to complex numbers and also mentions misconceptions that students may have in their understanding of complex numbers in both standard and polar forms.

## Comments about the Development of Concepts

One necessary concept for developing an understanding of the complex number system is to recognize the need for such a system from the perspective of both algebra and geometry. Depending on the historical perspective of mathematics curriculum development within a particular context, a curriculum developer could emphasize an algebraic approach or a geometric approach. However, students studying the properties of complex numbers using both algebraic and geometric approaches may develop a deeper understanding of the properties of complex numbers.

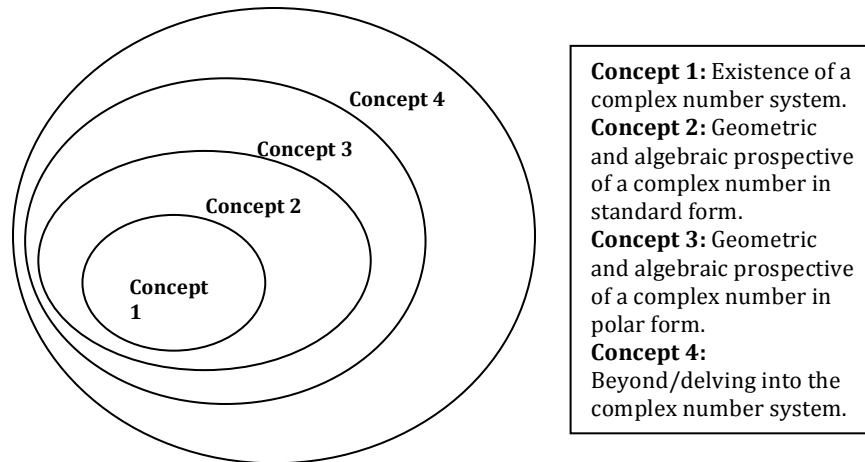
Students' understanding of both the algebraic and geometric approaches of complex numbers in standard form may be extended to the algebraic and geometric perspectives of complex numbers in polar form. When the study of complex numbers includes the polar form and de Moivre's Theorem, students have the opportunity to develop another perspective on geometrical representations of multiplication and division of complex numbers, finding the powers and the roots of complex numbers, rotations in a plane, and trigonometry. A further extension of the study of complex numbers in standard and polar forms examines the complex roots of a polynomial, algebraic structures or systems such as 'quaternion' as proposed by Hamilton in 1833.

## How might this brief be used?

This brief explicates the conceptual development of the study of complex numbers by examining three main concepts that could be included in the secondary school mathematics curriculum and suggests an additional concept that could be used for

further study. The concepts are tightly linked, and concept 1 should be studied prior to concept 2; concept 2 prior to concept 3; and concept 3 prior to concept 4.

Figure 1 illustrates ways in which the concepts related to an understanding of complex numbers are embedded.



**Figure 1**

**Concept 1: The Creation of a Complex Number System**

Students examine the creation of a complex number system beginning with solutions obtained from the process of solving equations; in particular complex numbers can be created to find solutions for a quadratic equation. The examples in Table 1 show how solutions to particular equations exist in different number systems before the complex number system is created.

**Table 1**

Solve the following equations:	Students may say:	Purpose of each equation:
a) $x + 1 = 0$	a) I cannot find a solution in the natural numbers. I should find the solution in the integers.	a) The solution is a negative integer
b) $3x-1=0$	b) I cannot find a solution in the integers. I should find the solution in the rational numbers.	b) The solution is a rational number
c) $x^2 - 5 = 0$	c) I cannot find a solution in the rational numbers. I should find the solution in the real numbers.	c) The solutions are irrational and real numbers

d) $x^2 + 4 = 0$	d) I cannot find a solution in the real numbers.	d) The solutions involve the square root of a negative number
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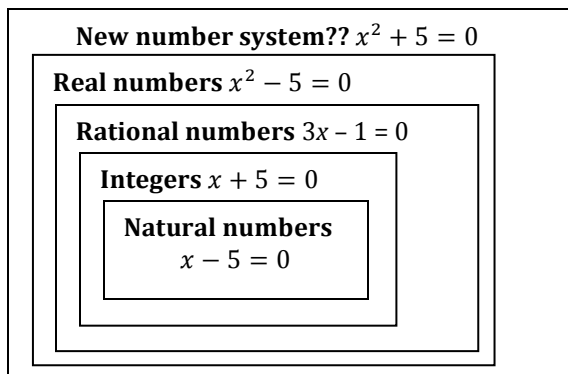
Thus, the creation of the set of complex numbers may start with the idea that there is no real solution for the  $\sqrt{n}$  when  $n$  is a negative number, i.e., there is no solution to the given equation in the real number system.

The need for a solution leads to describing the equation  $x^2 + 4 = 0$  as having imaginary solutions that cannot be represented in the set of real numbers. In the imaginary solutions,  $i$  is used where  $i^2 = -1$ , or  $i = \sqrt{-1}$ .

Writing the imaginary solutions of  $x^2 + 4 = 0$  is seen as follows:

$$\begin{aligned}
 x^2 + 4 &= 0 \\
 (x + \sqrt{-4})(x - \sqrt{-4}) &= 0 \\
 x &= \sqrt{-4}, \text{ or } x = -\sqrt{-4} \\
 x &= \sqrt{4(-1)}, \text{ or } x = -\sqrt{4(-1)} \\
 x &= 2\sqrt{-1}, \text{ or } x = -2\sqrt{-1} \\
 \text{Thus, } x &= 2i, \text{ or } x = -2i
 \end{aligned}$$

As shown in Figure 2, the solution of each equation in Table 1 led to a different number system.



**Figure 2**

At this point students know that an equation that has no solution in the set of real numbers may have one if imaginary numbers are used. The next step is to consider a quadratic function  $y = ax^2 + bx + c = 0$  ( $a \neq 0$ ) and how to find its zeroes) graphically and algebraically. Students should explore the 3 cases of a quadratic function:

- Quadratic function with two real zeroes, represented on a graph as intersecting the x-axis two times

- Quadratic function with one real zero represented on a graph as intersecting the  $x$ -axis one time
- Quadratic function with no real zeroes when the graph does not intersect the  $x$ -axis

### Concept 2: Geometric and Algebraic Perspectives of Complex Numbers

(For a more detailed explanation, see the brief *Integrating Algebra and Geometry with Complex Numbers*.)

A complex number  $z = a + bi$  has a real part  $a$ , and an imaginary part  $bi$ . The complex number  $z$  can be represented as an ordered pair  $(a, b)$  in the complex plane where the horizontal axis consists of real numbers,  $a$ , and the vertical axis represents imaginary numbers,  $bi$ . Each complex number corresponds to exactly one point on the complex plane, thus exhibiting a one-to-one correspondence between the set of complex numbers and the set of points in the plane.

Complex numbers are not ordered, but like the real number system, are closed under the four arithmetic operations of addition, subtraction, multiplication and division, with the exception of division by 0. The commutative, associative, distributive, identity and inverse properties hold with respect to addition and multiplication of complex numbers just as they do for the set of real numbers.

Some geometric aspects of complex number should also be examined:

- Conjugate  
The conjugate,  $a - bi$ , of a complex number,  $a + bi$ , is the reflection of  $a + bi$  in the  $x$ -axis.
- Absolute value  
The absolute value of a complex numbers is the distance between the origin and the point  $(a, b)$  in the complex plane.
- Addition  
The addition of two complex numbers can be thought of as vector addition because each complex number can be considered a vector.  
If  $z_1 = a + bi$  and  $z_2 = c + di$ , then  $z_1 + z_2$  is the translation of  $z_1$ ,  $c$  units horizontally and  $d$  units vertically.
- Multiplication
  1. If a complex number  $z = a + bi$  is multiplied by  $i$ , then
 
$$zi = (a + bi)i$$

$$zi = -b + ai$$

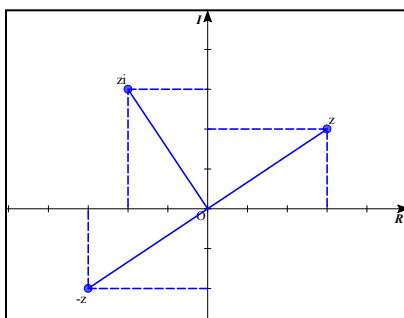
Graphically, the multiplication of a complex number,  $z$ , by  $i$  is a  $90^\circ$  rotation of point  $z$ .

Graphically, the multiplication of a complex number,  $z$ , by  $-i$  is a clockwise rotation of  $90^\circ$  of  $z$ .

2. Multiplying a complex number  $z = a + bi$  by  $(-1)$  results in  $z(-1) = -a - bi$ .

Graphically, the multiplication of a complex number,  $z$ , by  $(-1)$  is a  $180^\circ$  rotation of point  $z$  or the reflection of  $z$  in the origin.

A representation of  $z$ ,  $zi$ , and  $-z$  in the complex plane is seen in Figure 3.



**Figure 3**

The geometrical meaning of the multiplication of a complex number by  $i$  can be proved using a variety of methods including the use of slope and the scalar product of vectors. The slope method is demonstrated here:

The slope of the line through point  $z$  and the origin is  $m_1 = b/a$ , and the slope of the line through point  $zi$  and the origin is  $m_2 = -a/b$ .

$$m_1 \cdot m_2 = \frac{b}{a} \cdot \left(-\frac{a}{b}\right) = -1$$

Thus, the two lines are perpendicular showing that multiplication of a complex number by  $i$  results in a  $90^\circ$  rotation of the complex number.

3. Multiplication of a Complex Number by a Real Number

Students may explore multiplication of complex numbers by a real number as a dilation.

4. Multiplication of Two Complex Numbers

If  $z_1 = a + bi$  and  $z_2 = c + di$ , then  $z_1 z_2 = (a + bi)(c + di) = (ac - bd) + (bc + ad)i$ .

Geometrically, the product of two complex numbers can be found by multiplying their absolute values and computing the sum of the measures of the associated angles. (See *Integrating Algebra and Geometry with Complex Numbers*.)

### Concept 3: Complex Numbers with Polar Representation

A complex number can also be represented in a polar coordinate system; using trigonometry, a complex number  $z = x + iy$  is represented in polar form as  $z = |z|(\cos\alpha + i\sin\alpha)$ . When using polar form, students should be aware that it is not sufficient to express an argument  $\alpha$  of a complex number as  $\alpha = \arctan(y/x)$  because  $\arctan(y/x)$  gives only angles such that  $\alpha \in (-\pi/2, \pi/2)$  whereas  $\alpha \in (-\pi, \pi)$ . One way to determine the angle used is to plot the complex number in the complex plane. For example, if  $z = -1 - i$  is plotted in the complex plane, then the argument is  $-3\pi/4$  and not simply the angle obtained by  $\arctan(1)$ . The complex number  $z = -1 - i$  in polar form is  $z = \sqrt{2}(\cos(-3\pi/4) + i\sin(-3\pi/4))$ ,  $z = \sqrt{2}(\cos(5\pi/4) + i\sin(5\pi/4))$ , or any other representation where any integral multiple  $2\pi$  is added to the argument.

#### Multiplication of Complex Numbers in Polar Form

To multiply complex numbers in polar form, students must know the compound angle formulas for sine and cosine. To get the reciprocal of  $z$ , or  $|z|^{-1}(\cos(-\alpha) + i\sin(-\alpha))$ , students should know also that the cosine is an even function and the sine is an odd function. Additionally, students learn that when multiplying complex numbers, arguments are added and when dividing complex numbers, arguments are subtracted. These lead to a geometrical meaning of multiplication and division of complex numbers.

Multiplication of complex numbers in polar form is directly connected to de Moivre's theorem, which can be proved by mathematical induction for natural number values of  $n$ . Further, de Moivre's theorem is used to find roots and powers of complex numbers. The latter requires students to know the periodic properties of trigonometric functions.

Additionally, de Moivre's theorem can be used to express  $\cos(n\alpha)$  and  $\sin(n\alpha)$  as polynomials of  $\cos\alpha$  and  $\sin\alpha$  leading to other results in trigonometry.

### Concept 4: Delving into the Complex Number System

Advanced work in complex numbers may lead students to study the complex roots of a polynomial, algebraic structures, or systems such as quaternion as proposed by Hamilton in 1833.

#### Misconceptions

The study of complex numbers can create student misconceptions or difficulties.

Examples follow:

- $z = a + bi$  is a single number in the complex number system. A misconception is thinking of  $z$  as two numbers,  $a$  and  $bi$ , rather than as a single value that in and of itself has properties.
- $i$  is considered as a variable mainly because the calculation of  $(a + bi)(c + di)$  is accomplished as the multiplication of binomials.

- The quadratic formula works when the coefficients are complex numbers. A misconception is that the quadratic formula is only valid when the coefficients are real numbers.
- A single number has multiple representations. For example:  
 $1 + \sqrt{3} = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$ , but a misconception is that a single number has only one representation.
- $\sqrt{i}$  is a complex number that may be represented as  $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ . A misconception is that  $\sqrt{i}$  is not a complex number. It is also a misconception to think of  $\sqrt{i}$  as just one complex number. It is both  $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$  and  $-\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$