Conditions for the Effective Teaching and Learning of Reasoning and Proof

Reasoning and proof are essential processes in the doing and learning of mathematics. These two processes form an underlying part in the construction of new mathematical knowledge, both by the individual learner and in the mathematics community at large. In particular, the role of proof is multifaceted, including discovery, explanation, verification, communication and systematization (Mudaly, et al, 2004). Moreover, reasoning is an indispensable process for solving most problems in mathematics. For these reasons, teaching and learning in school mathematics should include reasoning and proof as integral elements.

Since learning is a social process in which students organize their actions through interactions with other students and with their teacher, it is fundamental to consider carefully the conditions that allow effective and meaningful teaching and learning of reasoning and proof. Learners go through a complex process of internalization and conceptualization that is deeply influenced by theircommunication with others. Thus the learning of reasoning and proof can be developed through a student-centered approach in which collaboration is essential. The aim is to provide students, from an early age, with a variety of experiences of reasoning and proof where they gradually develop the knowledge of reasoning and proof, with teacher scaffolding.

Some of the conditions necessary to support the teaching and learning of reasoning and proof are described in the following.

Curriculum

Since reasoning and proof are central to mathematics and mathematical learning we recommend that any mathematics curriculum represent reasoning and proof in a hierarchy of stages, through which students progress depending on their age, experiences and the culture in which they are studying. Thus students in different cultures will encounter proof at different times and in different ways. The earlier students begin to learn about reasoning, the stronger their understanding of proofs and proving will be.

The curriculum must emphasise the central nature of reasoning and proof with its presence in each of the content strands within the curriculum, including algebra, geometry, data and discrete mathematics. In addition reasoning and proof must be valued by inclusion in formative and summative assessment. The Park City Mathematics Institute brief *Assessment of Reasoning and Proof* describes several valid modes of assessment for reasoning and proof that go beyond asking for a traditional 'formal proof'. These modes of assessment are to be used in formative assessment and summative assessment, including internal and external examinations.

Material Resources

Students should have access to a flexible learning environment that allows for individual and collaborative learning. Ideally, students should have a safe, comfortable and well-equipped place to study, access to textbooks, writing equipment and a range of resources. These should include materials some form of investigative methodology, for example manipulatives, graphing calculators or computer hardware with appropriate mathematical software such as interactive geometry, statistical software, spreadsheets and graphing programs. Where access to such resources is limited teachers should collaborate in developing resources through the creative and imaginative use of available materials. For example, the Pythagorean Theorem can be explored by paper folding or specially designed manipulative tiles, pattern investigation, and ruler and compass constructions.

Human Resources

In order to manage the teaching and learning of reasoning and proof, teachers need to be aware of the role and function of reasoning and proof in mathematics, knowledge of mathematics appropriate to the level they are teaching, how to conduct mathematical investigations and what it means to work mathematically. It is important that teachers' skills include managing student-centered learning, competence in assessing reasoning and proof, and the ability to manage and assess investigative tasks as their students actively experience, engage, explore, conjecture, validate or refute, and formalize.

Gail Burrill 9/28/07 4:26 PM
Comment: What is experimental mathematics?

Teacher Education

Teaching and learning are culturally dependent. As a consequence teacher education in reasoning and proof will differ from country to country.

Regardless of the country, practicing teachers need to develop a culture of reflection, evaluation and renewal so that they can deliver relevant and effective teaching of reasoning and proof in their classrooms. Access to a vibrant, fertile and continuing discussion in the teaching community is vitally important. A central focus of the discussion should be on the latest and best in practice and content for collaborative learning, investigative tasks, and the development and testing of conjectures.

Pre-service teachers will become better practicing teachers if they experience the methods described above in their own education as teachers. Thus they should undertake open-ended investigative tasks that develop reasoning, work collaboratively to experience argumentation and the communication of ideas and come to an understanding of the axiomatic structures that are used to describe and to characterize, for example, Euclidean geometry and the structure of the real numbers. Pre-service teachers should use the opportunity presented by their teaching practice opportunities (professional stage) to implement these important strategies in the classes that they teach.

Teachers' use of investigative methodologies and materials, including technology

The most powerful ways of motivating proof involve providing students with an environment to make conjectures by themselves and some encouragement to systematically explore them, leading to a proof to solidify the conclusion. Some students are motivated when provided with situations where they have to predict and then determine whether their predictions are valid. The use of such open-ended situations and mathematical investigations are good ways of initiating such work, allowing students to engage in a form of mathematical experimentation. Opportunities to work within a small group increases the likelihood that students will be motivated to prove their own conjectures are correct, in order to persuade fellow students or classmates.

Computers and calculators have the potential to motivate proof as they provide new opportunities to experiment with mathematical ideas and objects, to detect patterns and regularities, leading to conjectures that require proof. Many teachers have reported on ways of using dynamic geometry systems, spreadsheets, calculators and other mathematical tools for such purposes.

Instructional materials, which could include the mentioned technology, should provide rich and engaging tasks that lead the students to reason, to conjecture and to communicate about the object of the investigation. Teachers should understand how to connect the mathematics with the investigation and be aware that it is possible for students to make the wrong assumptions in a given situation. The role of the teacher is pivotal in facilitating, monitoring and providing guidance in using these instructional materials. Examples of investigative tasks that could be used to foster reasoning and proof are given in the appendix.

Appendix

Examples of investigative tasks that foster reasoning and proof

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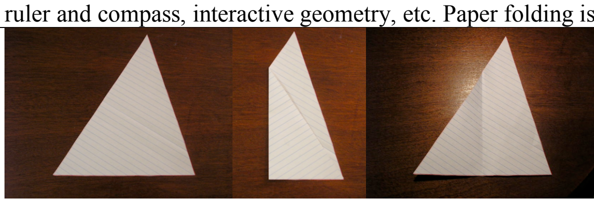
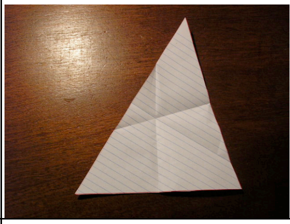
The following examples illuminate students' activities in terms of the learning process: experience, engage, explore, conjecture, validate or refute, and formalise.

Example 1: Algebra and Number

Learning process	Suggested guidelines for proceeding
Experience Students calculate the result of the two subtractions.	Calculate 987 654 - 789 -456
Engage Students get the same result.	What do you notice?
Explore Teacher initiates exploration and discussion. Several types of response are possible	Can you find some more examples like this? How are your new examples similar?
Conjecture Teacher facilitates conjecturing process	Can you explain why?
Validate or refute Students communicate about reasoning	How can you convince another student that your explanation is correct?
Formalise Teacher monitors, scaffolding if necessary.	How can you be sure, mathematically, that this will always be the case?
Application Critical application of new knowledge.	Can you find the pattern for numbers with 2, 4, 5, 6, . . . digits?

Example 2: Two-dimensional geometry

Learning Process	Suggested guidelines for proceeding: many modes are possible, including ruler and compass, interactive geometry, etc. Paper folding is illustrated.
Experience Create the perpendicular bisector of each side.	 Make a fold that is the perpendicular through the midpoint of a side of your paper triangle. Repeat for the other two sides.
Engage The perpendicular bisectors are concurrent.	 What do you notice about the three folds?
Explore Teacher initiates exploration and discussion.	Ask some other students if you can see what happened to their folds? What shape were their triangles? What do you notice about their three folds?

Conjecture Teacher facilitates conjecture.	Was there anything common in the folds in everybody's triangles? Can you explain your answer?
Validate or refute Teacher asks guiding questions.	What folding experiment might help to support your conjecture? Make another experiment with one fold. Does this give you any evidence? What about two folds? Does it help to mark the point where two folds cross?
Formalize To proceed to this stage the students would have experiences in Euclidean geometry (congruent triangles) or in Cartesian geometry (slope and equation of a line).	How could you use the ideas from your experiment and your knowledge of geometry to be sure, mathematically, that this will always be the case?
Application Critical application of new processes.	What happens if you use perpendicular ***tri***sectors, or ***angle*** bisectors? How do you justify the answers you give?

References

Mudaly, V., and de Villiers, M. "Mathematical Modeling and Proof." 2004.
Paper presented at 10th AMESA Congress, AMESA, 30 June -4 July 2004, University of the North-West, Potchefstroom. See http://www.amesa.org.za .