1 *IMCePtion*

Hey, welcome to the class. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized: s

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How did others at your table think about this question?
- **Respect everyone's views.** Remember that you have something to learn from everyone else. Remember that everyone works at a different pace.
- **Learn from others.** Give everyone the chance to discover, and look to those around you for new perspectives. Resist the urge to tell others the answers if they aren't ready to hear them yet. If you think it's a good time to teach everyone about manifolds, think again: the problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology: the problems should lead to appropriate uses of technology rather than requiring it. Try to avoid using technology to solve a problem "by itself". There is probably another, more interesting, way.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff, and maybe more. Check out Important Stuff first. The mathematics that is central to the course can be found and developed in Important Stuff. After all, it's Important Stuff. Everything else is just neat or tough. If you didn't get through the Important Stuff, we noticed... and that question will be seen again soon. Each problem set is based on what happened before it, in problems or discussions.

At least one problem in this course is unsolvable. Can you find them all?

Every three days, go back and read these again.

PROBLEM

Get Sketchpad working on your computer. The actual problem in the box will be done later.

Point your web browser to `http://www.tinyurl.com/getgsp`.

The first day of class is a perfect time for a huge logistical undertaking! Woo.

Important Stuff.

1. Sameer tells you the perimeter and area of a rectangle. Is it possible to confidently determine the dimensions of the rectangle?

Or whatever his name is. The hat guy.

2. Here are some perimeters and areas. Find the dimensions of each rectangle.
   (a) perimeter 24, area 36
   (b) perimeter 24, area 35
   (c) perimeter 24, area 32
   (d) perimeter 24, area 27
   (e) perimeter 24, area 34
   (f) perimeter 24, area 37

3. Find all solutions to each equation.
   (a) $x^2 - 12x + 36 = 0$
   (b) $x^2 - 12x + 35 = 0$
   (c) $x^2 - 12x + 32 = 0$
   (d) $x^2 - 12x + 27 = 0$
   (e) $x^2 - 12x + 34 = 0$
   (f) $x^2 - 12x + 37 = 0$

I found 'em! They were right over there!

4. Get Sketchpad on your computer!

5. A rectangle has perimeter 36. What could its area be?

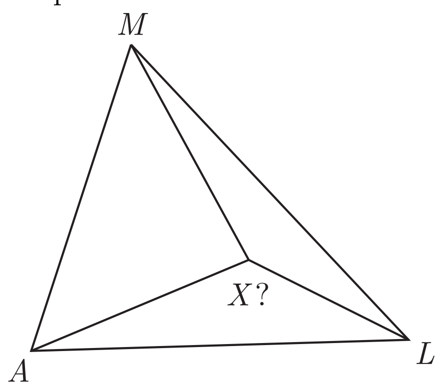
6. Chance tells you the surface area and volume of a rectangular box. Is it possible to confidently determine the dimensions of the box?

If you can confidently determine the dimensions, Chance will be sad: he'll have nothing left.

PROBLEM

Three cities located at points M, A, L get together to build an airport. Where should the airport be placed to minimize the lengths of the new roads that need to be built?

Three computers are located at points M, A, L. Where should a router be placed so that the smallest amount of cable is needed to connect it to the computers?



Build this sketch, then use Sketchpad to figure out where point X should be placed.

Is there anything special about this point?

There's a problem in the box *inside* the problem in the box! MIND-blowing. Let us know if you need help using Sketchpad to measure anything.

Neat Stuff.

7. For each point, decide if it is the same distance from $(7, 1)$ and $(-2, 9)$.

(a) $(7, 10)$
(b) $(-1, 1)$
(c) $(-17, -17)$
(d) $(-2, 0)$

This problem has four parts, but it's *not* multiple choice! Mind-blowing.

8. Find some rectangles whose perimeter and area have the same numeric value. More. MORE!

9. Find some rectangular boxes whose surface area and volume have the same numeric value. More?

10. Rina tells you the perimeter and area of a triangle. Is it possible to confidently determine the side lengths of the triangle?

11. Yesterday was 7/4/11, and $7 + 4 = 11$. Hooray for America. But...

(a) How many more times this century will there be a day like this? By *this* we mean the next one is August 3, 2011.

(b) How many times *next* century will there be a day like this?

(c) How can your second answer help you check the first?

At least one answer to this problem is totally awesome. Or found in the lyrics of a Talking Heads song. Or both.

Tough Stuff.

12. A triangle has perimeter 24. Find its maximum possible area, and explain how you know that this *must* be it.

13. Given positive integer n, the unit fraction $\frac{1}{n}$ can be written as the sum of two other unit fractions:

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

Find a rule for the number of ways to write $\frac{1}{n}$ as the sum of two unit fractions.

Like the blood type, a and b must be positive. Unlike the blood type, they must be integers.

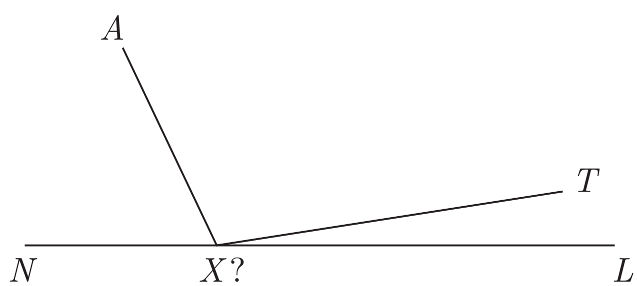
14. Find a rule for the number of ways to write $\frac{1}{n}$ as the sum of *three* unit fractions.

15. Spin a top so that it does not stop spinning. Or *does* it?

2 *PsyChoMetrIcs*

PROBLEM

Art (at point A) has to take a high-stakes test at point T. Since he is in Park City, he is extremely thirsty, and needs to take a big gulp from nearby river NL.



Where should he run toward (point X) to minimize the total distance $AX + XT$?

Drink water! Be sure your table synergies efficiently. If you figure out where point X should be, start thinking about how you could *construct* point X more directly, how you could *prove* that point X must be the right one, or how to change the problem in interesting ways.

What river is Art drinking from? deNiaL!

Okay, now it's time to think outside of the (rectangular) box.

Important Stuff.

1. (a) What's $(11 + \sqrt{2}) + (11 - \sqrt{2})$?
(b) What's $(11 + \sqrt{2})(11 - \sqrt{2})$?
(c) What's $(11 + x)(11 - x)$?

2. (a) A rectangle has perimeter 44 and area 117. What are its length and width?
(b) A rectangle has perimeter 44 and area 119. What are its length and width?
(c) A rectangle has perimeter 44 and area 100. What are its length and width?
(d) A rectangle has perimeter 44 and area 130. What are its length and width?

We mean a real rectangle here, not something as imaginary as a belief that Vanilla Sky is more popular than IMCePtion.

3. Find all solutions to each quadratic equation.

(a) $w^2 - 22w + 121 = 0$
(b) $w^2 - 22w + 120 = 0$
(c) $w^2 - 22w + 117 = 0$
(d) $w^2 - 22w + 119 = 0$
(e) $w^2 - 22w + 100 = 0$
(f) $w^2 - 22w + 2 = 0$
(g) $w^2 - 22w - 408 = 0$
(h) $w^2 - 22w + 130 = 0$

We like shortcuts, but we *especially* like shortcuts that do not involve the phrase "$4ac$".

4. Two numbers add up to 200 and their product is 9,991. What are the numbers?

5. Find three rectangles that have the same numeric value for their perimeter and area.

6. Blanca says she has found two rectangular boxes with different dimensions, but the same surface area and volume. Either find your own pair of boxes with this property, or prove that no such pair of boxes can exist.

These boxes show multiple personality disorder.

7. Triangle SAM has points $S(2, 1)$, $A(4, 1)$, and $M(4, 6)$.

(a) Draw triangle SAM in the plane.

(b) New points are created from the points in triangle SAM according to the rule

$$(x, y) \mapsto (-y, x)$$

Draw the new triangle created in this way in the same plane, and describe how this triangle is related to the original.

This Sam is not wearing a hat.

Check with others to verify the new points and feel validated.

8. Transform triangle SAM according to the rule

$$(x, y) \mapsto (-x, -y)$$

Draw the new triangle and describe how this triangle is related to the original.

We need you to leverage your enterprise. It's a paradigm shift.

9. (a) Show, beyond a reasonable doubt, that $(14, -4)$ is *not* equidistant from $(-2, -5)$ and $(4, 11)$.

(b) Again for $(-10, 7)$.

(c) Again for (x, y). Hmm!

Neat Stuff.

10. Cuong repeats the transformation in Problem 7 a whole bunch of times.

(a) What happens after the transformation is applied twice?

(b) What happens after the transformation is applied three times?

(c) . . . four times?

(d) . . . five times?

(e) . . . thirteen times?

(f) . . . 101 times?

These triangles won't self-actualize until you find their meaning and purpose.

11. For any rectangle you can assign a point (l, w) in a coordinate plane, defined by the length and width of the rectangle.

(a) Plot four points that all correspond to rectangles with area 20.

(b) How many rectangles are there with area 20? Plot them all.

(c) Plot all the rectangles with perimeter 20.

(d) Is there a rectangle with perimeter 20 *and* area 20?

12. (a) Put a point Q on the number line. Define $f(P)$ to be the distance from point P on the number line to point Q. What does the graph of $f(P)$ look like?

(b) Put point R on the number line, and now define $f(P)$ as the total distance from any point P to the two given points. What does the graph of $f(P)$ look like?

(c) Put point S on the number line, and do all that stuff we said again.

Sketchpad can be helpful here, but it's a challenge. You can pick specific values for the points, or be general.

These problems take a holistic approach in parabolizing the . . . yeah, we got nothin'.

13. Felipe says he has found two triangles with different side lengths, but the same perimeter and area. Either find your own pair of triangles with this property, or prove that no such pair of triangles can exist.

Careful, he might be delusional!

14. Find three rectangular boxes where the numeric value of their surface area equals the numeric value of their volume.

One of them is gleaming, like a 1989 skateboard movie.

15. Find three triangles that have the same numeric value for their perimeter and area.

16. Let $f(x) = x^3 + 3x^2 - 8x - 80$. Use long division to find the remainder when $f(x)$ is divided by each of these.

(a) $(x - 1)$

(b) $(x - 2)$

(c) $(x - 3)$

(d) $(x - 4)$

(e) $(x - 5)$

Feel free to uh, **divide** the work on this one among one another, there is no need to do all five of these yourself. But make sure you record all five answers and verify that the values are correct.

17. Complete this table for $f(x) = x^3 + 3x^2 - 8x - 80$.

x	$f(x)$
0	
1	
2	
3	
4	
5	

18. Find the remainder when $f(x) = x^{12} + 3x - 1$ is divided by each of these.
(a) $(x-1)$
(b) $(x-2)$
(c) $(x-10)$
(d) $(x+1)$

A codependent relationship with synthetic division will not help here.

19. Jessica takes triangle SAM from Problem 7 and applies a wacky transformation:

$$(x, y) \mapsto (x + y, -3x + 7y)$$

(a) Draw this new triangle JES. Is it even a triangle anymore?
(b) What is the area of this new shape? How does JES compare (in area) to SAM?

Jessica and Sam are equally-valued partners in our shared holism, and any comparison is purely for educational purposes.

20. Find all the ways to write $\frac{1}{10}$ as the sum of two unit fractions. Here's one for free:

$$\frac{1}{10} = \frac{1}{20} + \frac{1}{20}$$

Tough Stuff.

21. Given positive integer n, the unit fraction $\frac{1}{n}$ can be written as the sum of two other unit fractions:

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

Find a rule for the number of ways to write $\frac{1}{n}$ as the sum of two unit fractions.

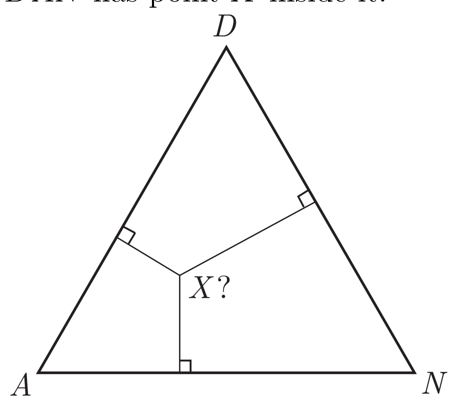
22. There's a point inside most triangles that forms three 120° angles with segments to the three vertices. A *Matsuura triangle* is a triangle whose side lengths are all integers, *and* whose three interior segment lengths from the 120° point are also integers. Find some Matsuura triangles, or prove they do not exist.

23. Find a rule for the number of ways to write $\frac{1}{n}$ as the sum of *three* unit fractions.

3 *The Ladybug PiCMIc*

PROBLEM

Equilateral triangle DAN has point X inside it.



Where should point X be placed so that the three *perpendicular* segments to the three sides of the triangle have a minimum total length?

So, the first task is to sketch a Dorito, sometimes called an "equilateral triangle"...

In Sketchpad, you can draw a perpendicular line... then what? I swear I once saw this picture on the outside of an Arby's in Hollywood.

Important Stuff.

1. Two numbers add up to 200, and their product is 9,919. Find the two numbers.

Look! Over there! Numbers!

2. (a) Two numbers add up to 30, and their product is 161. Find the two numbers.
(b) Two numbers add up to 30, and their product is $225 - n$. Find the two numbers.
(c) Two numbers add up to 30, and their product is 289. Find the two numbers.

Say, isn't 289 a perfect square? 8-15-17, or something like that? Oh, never mind.

3. For each point, determine whether or not it is 10 units away from $(13, -7)$.

(a) $(5, 3)$
(b) $(7, 1)$
(c) $(10, -16)$
(d) (x, y)

4. For any rectangle you can assign a point (l, w) in a coordinate plane, defined by the length and width of the rectangle. Rectangle $RAUL$ has length 10 and width 15.

Hope you brought some rectangular graham crackers to the PiCMIc.

(a) Plot four points that all correspond to rectangles similar to $RAUL$.
(b) How many rectangles are similar to $RAUL$? Plot them all.
(c) Find a rectangle similar to $RAUL$, but whose perimeter in units is larger than its area in square units.
(d) Find a rectangle similar to $RAUL$ whose perimeter and area have the same numerical value.

Two shapes are *similar* if one is a scaled copy of the other, like a napkin and a Cheez-It.

5. Darryl is so one-dimensional, he lives on the x-axis. He needs to make a round-trip once each to $(1, 0)$, $(3, 0)$, and $(17, 0)$ from home. Where should Darryl live to minimize his total travel distance? You have 30 seconds to guess. . . go!

How one-dimensional is he? His width and height are infinitesimal! *Ha!*

6. (a) Point Q has coordinates $(3, 0)$ and point P has coordinates $(x, 0)$. Define $f(P)$ to be the distance from point P to point Q. What does the graph of $f(P)$ look like?
(b) Point R has coordinates $(17, 0)$. Now define $f(P)$ as the sum of the distances PQ and PR. What does the graph of $f(P)$ look like?
(c) Point S has coordinates $(1, 0)$. You take it from here.
(d) Where should Darryl live to minimize his total travel distance?

OMG BBQ

7. Find three rectangular boxes where the numeric value of their surface area equals the numeric value of their volume.

One of them is a common shape of Jell-O.

8. Bill bought two briefcases this weekend. Remarkably, both have the same surface area and both have the same volume. And yet, the two briefcases are not the same size. Find some possible dimensions for Bill's briefcases.

However, one of the briefcases was 80% off. Ask Bill all about it. He'll be thrilled to tell you all the details.

9. Find two triangles where the numeric value of their perimeter equals the numeric value of their area.

Or not. There may not even be two such triangles. We ain't tellin.

Neat Stuff.

10. Find three pairs of positive numbers a and b, with $a \geq b$, that satisfy

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$$

11. Find all rectangles with integer side lengths whose area, in square units, is exactly double the perimeter in units.

12. Go back and do problems 16 through 18 from Day 2 if you haven't already. They are *cool*.

Problem 12 is the ladybugs' favorite.

13. Let $p(x) = x^4 - 2x^3 + 3$ and $q(x)$ is the *remainder* when $p(x)$ is divided by $(x-2)(x-5)\ldots$ in other words, $x^2 - 7x + 10$.

But Problem 13 is like ants at the PiCMIc.

(a) Complete this table of values for $p(x)$:

x	$p(x)$
0	
1	
2	
3	
4	
5	

(b) Complete this table of values for $q(x)$:

x	$q(x)$
0	
1	
2	
3	
4	
5	

Notice anything interesting?

14. Let $f(x) = 3x^2 - 10x + 21$. Use polynomial long division to find a linear function that agrees with $f(x)$ when $x = 3$ and when $x = -5$.

15. Let $f(x) = 3x^2 - 10x + 21$. Ooh, the same function.

(a) Use division to find $g(x)$, a linear function that agrees with $f(x)$ when $x = 3$ and when $x = 3$.

This isn't a typo. Use the method from Problem 14. *THIS ISN'T A TYPO!*

(b) Graph $f(x)$ and $g(x)$ on the same axes, using Sketchpad or a graphing calculator. What do you notice?

16. Find the equation of the tangent line to $f(x) = x^3$ at $x = 2$. Do *not* use the calculus!

17. Do the problem in the box, but this time use a generic triangle instead of an equilateral one. What happens?

Okay, who brought the generic-brand Doritos to the PiCMIc. . .

18. (a) Point A has coordinates $(0, 0)$ and point P has coordinates (x, y). Define $f(P)$ to be the distance PA. What does the graph of $f(P)$ look like?

(b) Point L has coordinates $(8, 0)$. Now define $f(P)$ as the sum of the distances PA and PL. What does the graph of $f(P)$ look like?

(c) Point M has coordinates $(3, 6)$. You take it from here.

19. In the box from Session 2, Art runs just as fast to the river as he does away from it. But in reality, Art runs twice as fast when he's thirsty as he does when he's not. How does this change the sketch and its solution? Can you model this in Sketchpad? On a graphing calculator? What is Snell's Law?

20. What is the largest prime number ever used in the lyrics of a Top 40 song?

Tough Stuff.

21. Find the equation of a parabola that is tangent to the function $f(x) = x^4 - 2x^3 + 3$ at $x = 1$, and also intersects $f(x)$ at $x = -1$. Under no circumstances are you to use the calculus for this problem! We'll know!

22. Given a positive integer k, there are a number of values of b so that the quadratic $x^2 + bx + kb$ is factorable over the integers. Determine, based on k, how many such values of b there are.

23. The quadratic equation $x^2 - 10x + 22 = 0$ has two roots.

(a) Find a quadratic whose roots are the *squares* of the roots of $x^2 - 10x + 22 = 0$.

(b) Find a quadratic whose roots are the nth powers of the roots of $x^2 - 10x + 22 = 0$.

24. Find all integer solutions to this system of equations:

Nope, there's more.

$$a + b = cd$$
$$c + d = ab$$

25. There's a point inside most triangles that forms three 120° angles with segments to the three vertices. A *Matsuura triangle* is a triangle whose side lengths are all integers, *and* whose three interior segment lengths from the 120° point are also integers. Find some Matsuura triangles, or prove they do not exist.

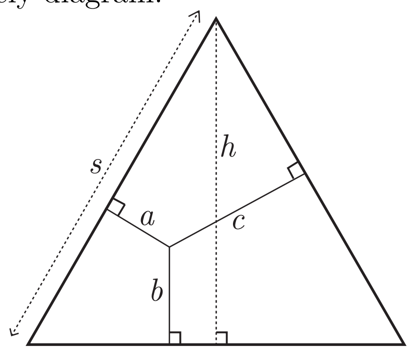
We'll keep these on the problem sets until someone gets 'em! Just kidding. Or are we...

4 *uPCoMIng attractions*

Important Stuff.

1. Look at this lovely diagram!

Like Roxette, it's got the look.



(a) Find an expression for the area of the equilateral triangle pictured above.

(b) Find a triangle with area $\frac{1}{2}as$. It might be hiding!

(c) Find a second expression for the area of the equilateral triangle, then work your magic.

It's okay for the expression to have more than one variable, just like it's okay not to have the box come first.

Poof goes the proof, and boom goes the dynamite.

PROBLEM

Sketch a right triangle that stays a right triangle when you move its points around. (What would you need to construct first?)

Then, build three equilateral triangles on the outside of the right triangle by using each side of the right triangle as the base for an equilateral triangle.

Find a relationship between the areas of the equilateral triangles.

We interrupt today's Important Stuff with this friendly and informative moment: the Pythagoreans were a bunch of squares.

PROBLEM

Redo the above construction using an arbitrary triangle (that is, a generic one instead of a right triangle) as the starter, building equilateral triangles along each side.

Use this sketch to *construct* the 120° point. See if you can figure out how, and why. If you've already built this construction, try to find another. The sketch from Day 3 might help!

2. Now that you are in new groups, ask your tablemates for the dimensions of some rectangles whose area and perimeter have the same numeric value. Compile a list. For each rectangle, calculate

$$\frac{1}{L} + \frac{1}{W}$$

where L and W are the length and width of each rectangle.

Gotta make my mind up... which seat can I take?

3. Rectangle $ARON$ has length 20 and width 15.

(a) In a coordinate plane, plot (l, R), where l is the length of $ARON$ and R is the ratio of the area of $ARON$ to the perimeter of $ARON$.

(b) Find several rectangles similar to $ARON$, and plot them on the same axes.

(c) Find the one rectangle similar to $ARON$ whose perimeter and area have the same numeric value.

4. Tom starts with a 3 by 4 by 5 box.

(a) Mimi finds a box similar to Tom's box whose surface area and volume have the same numeric value. What are the dimensions of Mimi's box?

(b) Call the dimensions of Mimi's box T, O, and M. Find the exact value of

$$\frac{1}{T} + \frac{1}{O} + \frac{1}{M}$$

What!!

$\frac{1}{O}$ is not quite the same as dividing by zero...

5. For each point, determine whether or not it is twice as far from $(20, 0)$ as it is from $(5, 0)$.

This one isn't multiple choice, either!

(a) $(10, 0)$
(b) $(-6, 8)$
(c) $(8, 6)$
(d) (x, y)

6. Two positive numbers multiply to 49. What is the largest and smallest sum possible?

7. Which of the following is the worst possible title we could have used this week?

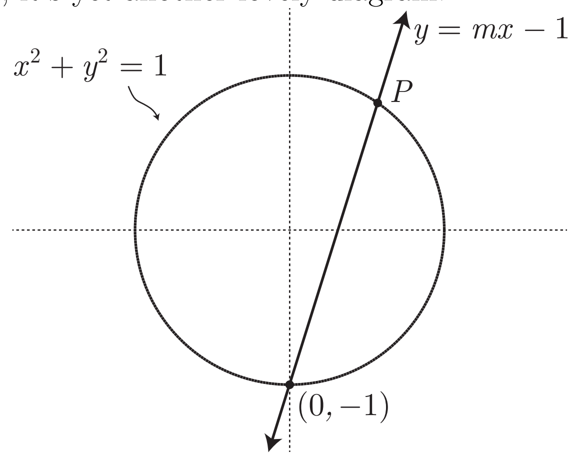
Write in part (j) and submit your own!

(a) hyPoglyCeMIa
(b) PsyChosoMatIc
(c) y-IMterCePt
(d) I, MuttonChoPs
(e) nIMCompooP
(f) I, creaMCuPs
(g) rePubliCaMIze
(h) I, MicroChiP
(i) PoliCyMakIng
(j)

Neat Stuff.

8. Look, it's yet another lovely diagram!

And it goes, "Na na na na na, na na na na na, na na NA na na na..." It's got the look. Sorry.



For each value of m given, determine the exact coordinates of point P.

(a) 2
(b) $\frac{3}{2}$
(c) $\frac{4}{3}$
(d) 4
(e) 10
(f) $\frac{a}{b}$

9. Let $f(x) = x^3 - 6x^2 + 4x + 8$. For each quadratic below, find the remainder when $f(x)$ is divided by the quadratic, then plot $f(x)$ and the remainder on the same axes (using techmology). What do you notice?

Sketchpad can do this too? Indeed.

(a) $(x-2)(x-6) = x^2 - 8x + 12$
(b) $(x-2)(x-5) = x^2 - 7x + 10$
(c) $(x-2)(x-4) = x^2 - 6x + 8$
(d) $(x-2)(x-3) = x^2 - 5x + 6$

10. Let $f(x) = x^3 - 6x^2 + 4x + 8$. Do what you did on the last problem. What do you notice?

(a) $(x-2)^2$
(b) $(x-3)^2$
(c) $(x-4)^2$
(d) $(x-5)^2$

11. Find the equation of the tangent line to $f(x) = x^4$ at $x = 1$.

12. Mary draws an equiangular octagon with alternating sides of length 3 and length 2. She then challenges you to find the point inside the octagon with the minimum total of the *eight* altitudes drawn from that point to the sides of the octagon. Go!

All the angles in this octagon are 136 degrees within ± 1 margin of error. As with triangles, some of the altitudes may extend outside the shape.

13. If you draw the altitudes from any interior point to all sides of an equilateral triangle, the altitudes' lengths add up to a constant. What about these shapes? For those that work, prove it; for those that don't, explain why they don't work.

(a) a square
(b) a rectangle
(c) a rhombus
(d) a parallelogram
(e) a regular pentagon
(f) an equilateral hexagon (not regular)
(g) an equiangular hexagon (not regular)

14. The sum of two numbers is s and the product is p. Find the sum of the...
(a) squares of the two numbers.
(b) cubes of the two numbers.
(c) fourth powers of the two numbers.
(d) ... a generalization?

I got this, you got this... Now you know it!

Tough Stuff.

15. Take a triangle, and move its points according to the rule $(x, y) \mapsto (ax + by, cx + dy)$. Find integer values for a, b, c, and d so that the new shape has a smaller area than the original shape, but still *some* area.

16. So we've discovered that this 120° point gives the least possible total distance to the three vertices. But what about other points? They're worse, but some are not much worse. Indeed, the shape of the points that are *equally bad* is interesting. What's it look like? What's it look like if you move outside the original triangle?

17. Find several triangles that have integer side lengths (with no common factors) and a 120° angle. Generalize?

18. Find some Matsuura triangles, or prove they do not exist. (See previous sessions for the definition.)

19. Find this sum exactly:

$$0 + \frac{1}{100} + \frac{4}{10000} + \frac{9}{1000000} + \cdots + \frac{n^2}{10^{2n}} + \cdots$$

5 *Week 2: The Slurpe-NING*

Hey, welcome back to the class. We know you'll continue to learn a lot of mathematics here—some new tricks, some new perspectives on things you might already know about. A few things to recall about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How did others at your table think about this question?
- **Respect everyone's views.** Remember that you have something to learn from everyone else. Remember that everyone works at a different pace.
- **Learn from others.** Give everyone the chance to discover, and look to those around you for new perspectives. Resist the urge to tell others the answers if they aren't ready to hear them yet. If you think it's a good time to teach everyone semiperimeter formulas, think again: problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology: the problems should lead to appropriate uses of technology rather than requiring it. Try to avoid using technology to solve a problem "by itself". There is probably another, more interesting, way.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff, and maybe more. Check out Important Stuff first. The mathematics that is central to the course can be found and developed in Important Stuff. After all, it's Important Stuff. Everything else is just neat or tough. If you didn't get through the Important Stuff, we noticed... and that question will be seen again soon. Each problem set is based on what happened before it, in problems or discussions.

Today is FREE SLURPEE DAY! Seriously.

Important Stuff.

1. Two positive numbers multiply to 81. What is the largest and smallest sum possible?

PROBLEM

Start a new Sketchpad sketch and follow these steps.

- Draw a circle.
- Draw a line that intersects the circle using two points that are outside the circle, and on opposite sides. Label one of these outside points A.
- Select the line and the circle, then click on "Construct" and "Intersection."
- Label the constructed points P and Q.

1. Find the location of Q that maximizes the sum $AP + AQ$. You can drag the other unlabeled point on the line to move P and Q. ♡
2. Find the location of Q that maximizes the product $AP \cdot AQ$.
3. Find the location of Q that minimizes the sum $AP + AQ$.
4. Move point A so that it is inside the circle. For this new location of A, find the location of Q that maximizes the product $AP \cdot AQ$.

Say, this box is nearly a square. Circle gets the square! Paul Lynde would be proud, or at least snickering.

Two, two, two intersections in one!

We ♡ math! We ♠ our dog! We ♣ all weekend! We ♢ a shrubbery!

2. Find a rectangle similar to a 7 by 11 rectangle and has the same numerical value for area and perimeter.

3. Todd starts with a 5 by 6 by 20 box.

(a) Betul finds a box similar to Todd's box whose surface area and volume have the same numeric value. What are the dimensions of Betul's box?

(b) Call the dimensions of Betul's box T, O, and D. Find the exact value of

$$\frac{1}{T} + \frac{1}{O} + \frac{1}{D}$$

We were *this* close to putting a 7 by 11 by 61.6 box on the problem set instead. Oops, we just did!

4. For each of these points, determine whether or not it is three times as far from $(9,0)$ as it is from $(1,0)$.

(a) $(0,3)$
(b) $(7,0)$
(c) $(-2,2)$
(d) $(-3,0)$
(e) $(1,-5)$
(f) (x,y)

5. Do the "*SAM*" problem (number 7) from Day 2 if you haven't already.

6. Let $s = 2 + i$ and $m = 4 + 6i$ be complex numbers. Find each of these.

(a) $s + m$
(b) $s - m$
(c) $3s + 3m$
(d) sm (the product)
(e) is
(f) im

When you see i^2, make it -1. The number i is the imaginary square root of -1.

What *is* it? That depends on the definition of *is*.

7. Simplify these seemingly nasty-looking expressions that involve square roots of square roots of negative numbers.

(a) $\sqrt{(3+4i)(3-4i)}$
(b) $\sqrt{(5-12i)(5+12i)}$
(c) $\sqrt{(15+8i)(15-8i)}$
(d) $\sqrt{(x+yi)(x-yi)}$

8. How far is each of these points from the origin?

(a) $(3,4)$
(b) $(5,-12)$
(c) $(15,8)$
(d) (x,y)

The coordinates of the origin are hole numbers.

9. (a) Draw a graph of all the points (x,y) that are 5 units from the origin.
(b) Write an equation for the graph you just drew.

10. Find all 12 complex numbers $a + bi$ with integers a, b so that

$$\sqrt{(a+bi)(a-bi)} = 5$$

A complex number with integers a, b is called a *Gaussian integer*. You can look up one of the 12 answers.

Useless Stuff.

11. What is $6 \div 2(1+2)$?

12. Solve for X:

$$(X^{10} + (\text{mushroom})^2)^4 + X^{10} = \text{snake}$$

"In the Middle Ages, people in convents were not allowed to eat beans because they believed something about them we now know isn't true. What?" Paul Lynde: "Well, I know they took a vow of silence..."

Today is FREE SLURPEE DAY! Seriously.

Neat Stuff.

13. For each point P, determine if it is twice as far from the *line* $x = 2$ as it is from the point $(-1, 3)$.

(a) $(5, 3)$
(b) $(0, 3)$
(c) $(-2, 4)$
(d) $(-3, 2)$
(e) $(-2, b)$
(f) (x, y)

Is $x = 2$ vertical or horizontal? Decide by asking what points are on its graph.

14. Square $ZACK$ has one vertex at $A(1, 0)$. The center of the square is the origin $O(0, 0)$. Find the coordinates of the other vertices.

15. Find all four solutions to the equation $x^4 - 1 = 0$.

Here's a useless reminder that $a^2 - 9 = (a+3)(a-3)$. Or *is* it?

16. Equilateral triangle DEB has one vertex at $B(1, 0)$. The center of the triangle is the origin $O(0, 0)$. Find the coordinates of the other vertices.

17. Find all three solutions to the equation $x^3 - 1 = 0$.

You can factor something by using long division. Divide, and if the remainder is zero, woot.

18. Regular hexagon $BUSHRA$ has one vertex at $A(1, 0)$. The center of the hexagon is the origin $O(0, 0)$. Find the coordinates of the other vertices.

19. Find all six solutions to the equation $x^6 - 1 = 0$.

20. Let $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. Take powers of z until it becomes awesome, then tell us the value of z^{101}.

The value of z^{101} is contributing awesome rock to Niagara Falls and the surrounding area!

21. Find a complex number z so that $z^{12} = 1$ but no smaller positive integer n has $z^n = 1$. Did we say "a"? We meant "all" of them.

More Useless Stuff.

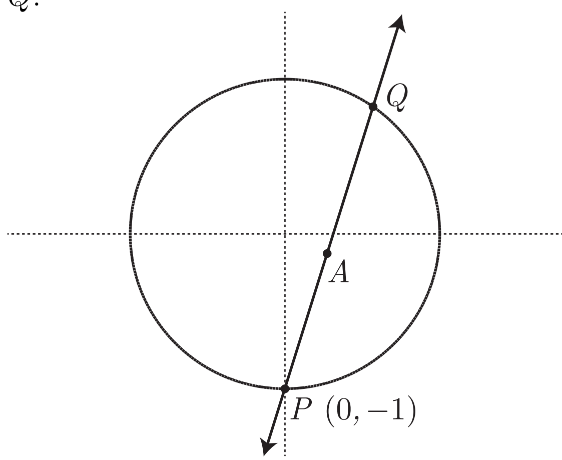
22. A free Slurpee is a truncated cone with a volume of 7.11 fluid ounces. Determine its total surface area.

23. Seriously, go get a free Slurpee!

"In one state, you can deduct \$5 from a traffic ticket if you show the officer...what?" Paul Lynde: "A ten dollar bill."

Neat Stuff.

24. Let P be a point at $(0, -1)$ and A be a point inside the unit circle. Let the line AP intersect the unit circle at point Q.



Calculate $f(A) = AP \cdot AQ$ for each of these points by determining the lengths AP and AQ.

(a) $A = (\frac{1}{2}, 0)$
(b) $A = (\frac{1}{4}, 0)$
(c) $A = (-\frac{1}{2}, 0)$
(d) $A = (\frac{1}{3}, -\frac{1}{2})$
(e) $A = (\frac{1}{3}, \frac{1}{2})$
(f) $A = (x, y)$

What happens if A is outside the circle?

"Why do sheep sleep huddled up?" Paul Lynde: "Because Little Boy Blue's a weirdo!"

Tough Stuff.

25. Suppose n is a positive integer. Find a rule that determines whether or not there is a right triangle with area n and rational numbers as side lengths.

26. Complete this long division problem, where all the missing digits are marked with an X. (There is no remainder.)

```
              X X 8 X X
X X X ) X X X X X X X X
          X X X
        -----
            X X X X
              X X X
            -----
                X X X X
                X X X X
                -------
```

PCMI rocks! Woot woot!

6 *Day of Recko-NING*

PROBLEM

Graph the circle $x^2 + y^2 = 65$ and determine all of its lattice points. A *lattice point* has integer coordinates.

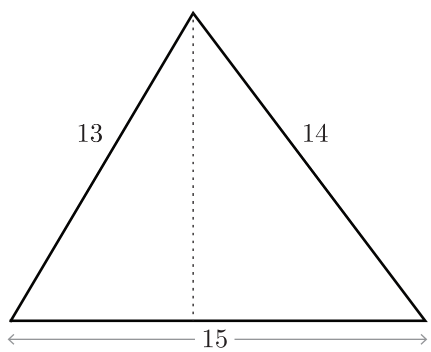
Please don't confuse this with a *lettuce point*, usually found at a salad bar.

Important Stuff.

1. Solve for h and x.

$$\begin{aligned} h^2 + x^2 &= 13^2 \\ h^2 + (15 - x)^2 &= 14^2 \end{aligned}$$

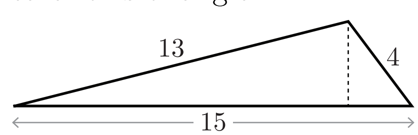
2. Find the perimeter and area of this triangle.



Bill says he will cut you if he sees you using anything with the letter s when solving this problem. The 13-14-15 triangle is called *Superheronian*! It can leap tall buildings in a single bound, but also its side lengths are consecutive integers and its area is an integer.

3. Find the side lengths of a triangle similar to the 13-14-15 triangle whose perimeter and area are equal numerically.

4. Find the area of this triangle.



In no particular order, Jack Bauer, Douglas Adams, Adele, and a calculator appreciate the answers to these triangle problems.

5. The *conjugate* of the complex number $z = a + bi$ is $\overline{z} = a - bi$.

(a) If $z = 5 + 2i$, what is $\overline{z}$?

(b) Let $w = 3 - 4i$. Calculate $w + \overline{w}$ and $w\overline{w}$.

(c) Find a complex number v so that $v + \overline{v} = 14$.

(d) Find a complex number v so that $v\overline{v} = 65$.

(e) Find a complex number v so that $v + \overline{v} = 14$ *and* $v\overline{v} = 65$.

By the way, $w\overline{w}$ is just w multiplied by $\overline{w}$.

6. Find two numbers whose sum is 14 and product is 65.

We're told you can solve this problem by "looking it up".

7. The *magnitude* of the complex number $z = a + bi$ is $|z| = \sqrt{z\bar{z}}$.

(a) Find the magnitude of $z = 5 + 2i$ and of $w = 3 - 4i$.

(b) Rewrite the equation $|z| = \sqrt{(a + bi)(a - bi)}$ as something that doesn't have i in it.

(c) Can the magnitude of a complex number ever be zero? Can it be negative?

(d) Find a complex number whose magnitude is $\sqrt{65}$.

Does anyone else remember the "Day of Reckoning" space from the Game of Life? That game has a spinner because dice are evil. Seriously: the original version came out in 1860 and had a spinning top with the numbers 1 to 6 on it, because dice are evil. The... More... You... Know!

8. How many complex numbers $a + bi$ have integer a, b and magnitude $\sqrt{65}$?

9. (a) Find the magnitude of $5 + 2i$.

(b) Find the magnitude of $(5 + 2i)^2$.

(c) Find a Pythagorean triple with hypotenuse 29.

10. Find a Pythagorean triple with hypotenuse 65.

11. Three positive integers add up to 25 and multiply to 360. What are the numbers?

Man, 360 is really divisible! Wouldn't it be awful if there were multiple answers here? Just awful. Seriously.

12. Find the volume and surface area of the box with dimensions $\frac{1}{15}$, $\frac{1}{6}$ and $\frac{1}{4}$.

Neat Stuff.

13. Consider the transformation rule

$$(x, y) \mapsto (5x - 2y, 2x + 5y)$$

For each point, calculate the new point created by this transformation rule.

(a) $P(1, 0)$

(b) $Q(5, 2)$

(c) $R(21, 20)$

(d) $S(65, 142)$

14. The notation $|z|$ that is used for magnitude is the same symbol as the notation used for absolute value. Do real numbers have the same magnitude as their absolute value?

Real numbers *are* complex numbers, they're just numbers like $7 + 0i$ or $-\sqrt{2} + 0i$.

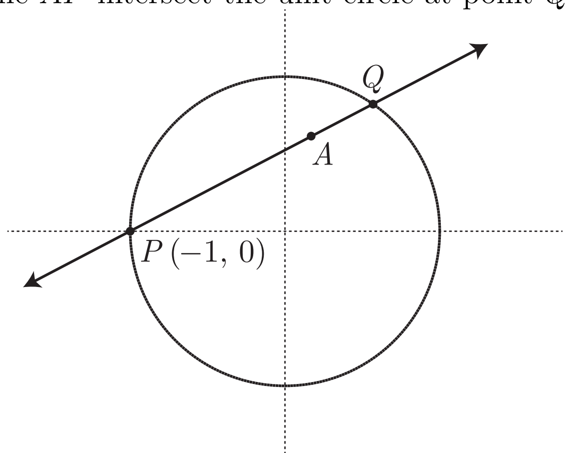
15. A *primitive Pythagorean triple* is a set of three positive integers a, b, c with $a^2 + b^2 = c^2$ where a, b, c share no common factors besides the blatantly obvious.

(a) Find a primitive triple with hypotenuse 65.

(b) Find *both* primitive triples with hypotenuse 85.

6, 8, 10 is *not* one of these, nor is 60, 80, 100.

16. Let P be a point at $(-1, 0)$ and A be another point. Let the line AP intersect the unit circle at point Q.



Calculate $f(A) = AP \cdot AQ$ for each of these points by determining the lengths AP and AQ.

(a) $A = (\frac{1}{2}, 0)$
(b) $A = (-\frac{1}{2}, 0)$
(c) $A = (\frac{1}{3}, -\frac{1}{2})$
(d) $A = (3, 2)$
(e) $A = (-3, 2)$
(f) $A = (x, y)$

17. Find a way to make triangles that have integer side lengths and integer area, and make a bunch of them. *No formulas!*

Bill mad! Bill smash if you use square roots or cosine.

18. Find more Superheronian triangles. Heck, find them all! There's a really small one, but 13-14-15 is the second-smallest.

Useless Stuff.

19. Solve for X:

Honey X don't care!

$$(X^{10} + (\text{mushroom})^2)^4 + X^{10} = \text{snake}$$

20. Solve for Y:

This band inspired today's problem in the box. Or *did* they? No.

$$((\text{I'm}) + Y + (\text{daba}(\text{dee} + \text{di}))^7)^2 = \text{Europop}$$

Tough Stuff.

21. Find all the triangles with integer-length sides whose area and perimeter have the same numerical value. Do they have anything else in common?

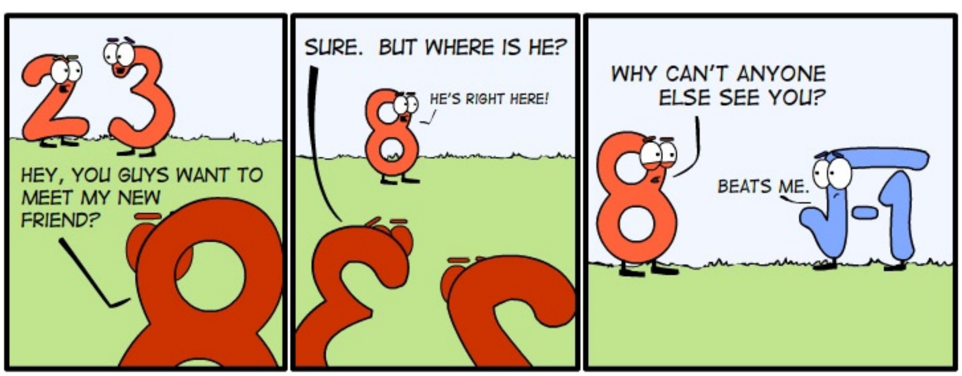
22. Find a number n that is the hypotenuse of exactly *four* primitive Pythagorean triples.

23. Same as the last one, but now exactly *eight* primitive Pythagorean triples!

24. Describe what types of positive integers n can be written in the form $n = a^2 + b^2$ for integer a, b. For example, 450 can be written this way: $450 = 21^2 + 3^2$.

25. The number 450 has eight odd factors: 1, 3, 5, 9, 15, 25, 45, 75, 225. Odd factors can be split into factors in the form $4k+1$ and $4k+3$. 450 has 6 odd $4k+1$ factors and 2 odd $4k+3$ factors. It turns out there's a remarkable connection between the number of these types of factors and the *number of different ways* 450 (or any positive integer) can be written in the form $n = a^2 + b^2$ for integer a, b. 450 can be written many different ways ($a = 3, b = 21$ is one of them). Figure out what the rule is—then prove that it works (harder). Don't forget that a or b can be negative or zero.

26. Nobody's found a Matsuura triangle yet. Maybe they don't exist.



by Thomas Dobrosielski `http://nfccomic.com/index.php?comic=205`

7 *Boom Win-NING*

PROBLEM

Pour some salt on some triangles... Umm... Yeah, we'll explain what to do.

Discuss the answers to these questions with your group. Feel free to use this GSP file to investigate:

`http://www.tinyurl.com/saltgsp`

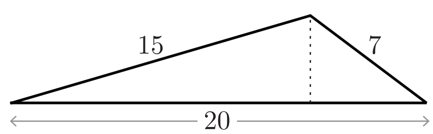
- What is so special about where the top of the salt heap is located?
- What is so special about where the ridges are located?
- Why is the top of the salt heap the same ________ to the three ______?

Today's problem in the box is a real happe-NING brought to us by our very own Troy Jones. Pour some salt on me! One more, in the name of math.

Charles Nelson Reilly would have the definitive answer here.

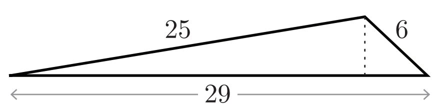
Important Stuff.

1. Find the area of this triangle.



This could be called a Boxx problem, or a Rapinoe problem, but most people will probably call it a Wambach problem.

2. Find the area of this triangle.



PROBLEM

Snag this file `http://www.tinyurl.com/circlegsp` and adjust the circle until it is the largest one that doesn't pass outside the triangle. What is the radius of this circle?

3. (a) In your diagram from the second box, find a triangle with area 20. It might be hiding!

(b) Find an expression for the area of the entire triangle in terms of things that aren't the height of the triangle.

A look back at problem Solo from day Sauerbrunn might be helpful.

4. Consider the complex numbers $s = 2 + i, a = 4 + i, m = 4 + 6i$.

(a) Plot and label s, a, and m in a complex plane.

(b) Multiply each number by i, then plot and label each of the new numbers in the *same* complex plane.

(c) Multiply each of s, a, and m by i *twice*, then plot and label each of the new numbers in the same plane.

(d) Three times? Four times? Five times? Thirteen times? 101 times?

Plot the complex number $x + yi$ at the point (x, y) in the *complex plane*. Don't worry, there are no snakes.

This all seems so familiar somehow. . .

5. Let $z = 1 + i$. Plot each of these in the same complex plane, and find the magnitude of each.

(a) z

(b) z^2

(c) z^3

(d) z^4

(e) z^5

6. Craig stands at the origin $(0, 0)$ and stares at the powers of $1 + i$ as they are built. Describe what happens to the powers from his perspective: where do they go? how far away?

7. Let $z = 3+2i$. Find the *magnitude* of each of these complex numbers.

(a) z

(b) z^2

(c) z^3

(d) z^4

z = Rampone + M$i$tts. But z should really just pick a value and stick to it. Y U NO STOP CHANGING Z!

8. Let $z = \frac{3}{5} + \frac{4}{5}i$. Plot and label each of these on the same complex plane. Approximations are fine here.

(a) z

(b) z^2

(c) z^3

(d) z^4

(e) z^5

9. Mahen stands at the origin $(0, 0)$ and stares at the powers of $\frac{3}{5} + \frac{4}{5}i$ as they are built. Describe what happens as accurately as you can. How is it similar to what happens with the powers of $1 + i$? How is it different?

Fortunately, Mahen is not a Care Bear, otherwise we'd all be in trouble.

Neat Stuff.

10. If $a = 3 + 4i$ and $b = 5 + 12i$, find the magnitude of a, the magnitude of b, the magnitude of ab.

11. (a) Find the magnitude of $w = 2 + i$.
(b) Perform an operation to w that results in a complex number that has magnitude 5.

Yes, we mean 5, not $\sqrt{5}$.

12. Do stuff to take each of these complex numbers and produce a primitive Pythagorean triple.
(a) $4 + i$
(b) $8 + 3i$
(c) $15 + 4i$
(d) $16 + 7i$
(e) $23 + 2i$
(f) $42 + 9i$

Triples are fun! This is Hurley's favorite problem. Hurley was so 2008. Boom. Boom. Pow. It's okay, we don't get it either.

13. (a) Three numbers add up to 14 and multiply to 72. Find both sets of three positive integers that work here.
(b) Use the results to find two boxes whose surface area and volume equal one another.

That's three numbers, not two and a half numbers. Boom.

14. Can two non-congruent triangles have the same perimeter and area? Explain!!

15. A triangle has side lengths a, b, and c. Without using fancypants formulas, find its height. What, there's more than one height? Fine, the area then.

Guys like me from Wall Street wear fancypants. Boom.

16. The magnitude of $3 + 4i$ is 5. Find *all* the Pythagorean triples with hypotenuse 125 (including non-primitive ones).

17. Use the concepts from problem 10 to find a primitive Pythagorean triple with hypotenuse 1105. Then another one! How many *are* there??

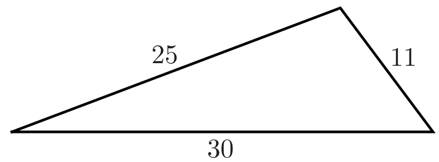
What's so special about 1105 anyway?

Tough Stuff.

18. (a) Find a way to generate all of the Pythagorean triples in which *the two leg lengths* are one away from each other. One example is 21, 20, 29.
(deux) Find a way to generate Pythagorean triples that are incredibly close to being 30-60-90 right triangles. They can't be, but they can be super close!

These problems are for the Hot Shots...

19. Besides being Heronian, this triangle has some other interesting feature. Find a way to generate more that have its special property.



20. Where are our Superheronian triangles! Bah!

21. $\frac{A}{BC} + \frac{D}{EF} + \frac{G}{HI} = 1$. Here, each letter is a unique and distinct number from 1 to 9, and the denominators are two-digit numbers. Find a solution!

Wait, are you reading this before doing the problems at the beginning? Get back to the first page, *now*!

22. As you keep taking powers of $z = \frac{3}{5} + \frac{4}{5}i$, will they eventually wrap around onto themselves? In other words, are there powers k and m with $z^k = z^m$ for this z?

23. In triangle ABC, angle A is twice angle B, angle C is obtuse, and the three side lengths a, b, and c are integers. Determine, with proof, the minimum possible perimeter.

8 *Pattern Sniffe-NING*

PROBLEM

For each of these complex numbers, plot the number and its square in the same complex plane. You might want to draw segments from the origin to each number. If you need to measure lengths or angles, consider using Sketchpad.

$$\begin{aligned} A &= 3+i \\ S &= 1+2i \\ H &= 3+2i \\ L &= \tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i \\ I &= i \end{aligned}$$

Keep doing examples until you can describe *precisely* where the square is located in relation to the original number.

Reminder: direct teaching is discouraged. All activities here can be completed without formulas.

You can type "3^0.5" to get $\sqrt{3}$ in Sketchpad.

Extension questions: what about cubes? square roots? reciprocals?

Important Stuff.

1. Let $z = \frac{12}{13} + \frac{5}{13}i$. Find and plot all of these on the same complex plane. (Estimate to three decimal places if you like.)

(m)	z^0	(n)	z^2	(c)	z^4
(o)	z^1	(i)	z^3	(a)	z^5

2. Use Sketchpad to find the *angle* that forms when you go from one power of $z = \frac{12}{13} + \frac{5}{13}i$ to the next.

You mean it's the same angle each time? Maybe, maybe not! You'll have to be a pattern sniffer, man.

3. What does it look like in the complex plane when you add two complex numbers w and z? Give some examples. What does $w + w$ look like?

We have clearance, Clarence. Roger, Roger. What's our vector, Victor?

4. If a complex number w is multiplied by a real number c (also called a *scalar*), what happens to the magnitude? the direction? What if c is negative?

5. If a complex number w is multiplied by i, what happens to the magnitude? the direction?

We heard that on Tuesday one participant multiplied their underwear by i^2.

6. Let $w = 1 + i$ and $z = \sqrt{3} + i$. Find the magnitude and direction of wz, and compare the results to the magnitude and direction of w and z.

7. What, in general, happens to magnitude and direction when you multiply two complex numbers?

Well, you know, the numbers, like, *combine*, man...

8. (a) Find the magnitude of $3 + 2i$.
(b) Calculate $(3 + 2i)^2$ and find its magnitude.
(c) What Pythagorean triple has hypotenuse 13?

9. Make some more Pythagorean triples. Totally awesum!!!

Neat Stuff.

10. (a) Three numbers add to 13 and multiply to 36. Find both sets of positive integers.
(b) Find two distinct rectangular boxes with volume 162 and surface area 234.
(c) If each box's dimensions are a by b by c, compute the product

$$(x - a)(x - b)(x - c)$$

and discuss any findings.

11. (a) Point Q has coordinates $(3, 0)$ and point P has coordinates $(x, 0)$. Define $f(P)$ to be the square of the distance from point P to point Q. What does the graph of $f(P)$ look like?

It looks like a big smile, man.

(b) Point R has coordinates $(17, 0)$. Now define $f(P)$ as the sum of the squares of the distances PQ and PR. What does the graph of $f(P)$ look like?
(c) Point S has coordinates $(1, 0)$. You take it from here.
(d) If $f(P)$ is the sum of the squares of the distances PQ, PR and PS, what P produces the minimum of $f(P)$?

12. For each point P, determine if it is twice as far from the point $(-1, 4)$ as it is from the *line* $y = 1$.

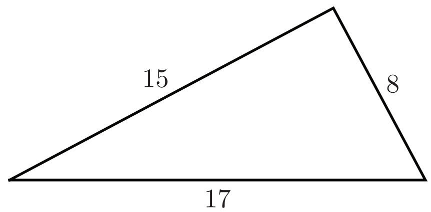
Is $y = 1$ vertical or horizontal? It depends on which way you've decided to graph, man.

(a) $(-1, 2)$
(b) $(-1, -2)$
(c) $(23, 14)$
(d) $(23, -14)$
(e) $(-91, 52)$
(f) (x, y)

13. Find an equation for all the points *three times* as far from $(-1, 4)$ as from the line $y = 1$.

14. Find an equation for all the points *half* as far from $(-1, 4)$ as from the line $y = 1$.

15. Find a triangle similar to the one below that has the same numerical value for its area and perimeter.



16. Find two triangles that have the same area and the same perimeter but that have dissimilar shapes.

17. A right triangle has leg lengths a and b. Find the radius of its incircle.

Try it with numbers first perhaps you will?

18. Use the results from problems 3-5 to prove that when you multiply a complex number by $a + bi$, it ...oh, crap, we forgot. What does it do?

19. Find a primitive Pythagorean triple whose hypotenuse length is 13^3.

What was that movie where Sean Astin finds a primitive Pythagorean triple in his backyard then Pauly Shore gives it a makeover? Oh, right, *Encino Math*.

20. Show how squaring the complex number $m + ni$ can be used to generate this identity that can be used to generate Pythagorean triples:

$$\left(m^2 - n^2\right)^2 + \left(2mn\right)^2 = \left(m^2 + n^2\right)^2$$

21. Let a be a complex number with magnitude 5, and b be a complex number with magnitude 13. Consider $a+b$: what could its magnitude be? Do you add magnitudes when you add complex numbers, or what?

Vegeta says these magnitudes can be over 9000!!!

22. Suppose x can be written as the sum of two squares, and y can also be written as the sum of two squares. Prove that xy can *also* be written as the sum of two squares.

23. As the powers of $z = \frac{12}{13} + \frac{5}{13}i$ grow, eventually the powers work around the four quadrants, then back around into Quadrant I. Find the first power of z that re-enters Quadrant I.

The powers of z are movin'! They're numbers ...on the *grow*, man.

24. Suppose a regular n-gon is placed with its center at the origin and one of its vertices at $(1, 0)$.

(a) For what values of n will $(-1, 0)$ be a vertex of the n-gon?

(b) For what values of n will $(0, -1)$ be a vertex of the n-gon?

(c) Find all points that are on the regular 12-gon but aren't on any smaller n-gon.

Darryl says the rectangles in Cracklin' Oat Bran are regular n-gons.

25. Take all the vertices on the regular 12-gon as points in the complex plane. Square them and plot the squares. What happens?

26. You've now seen that squaring complex numbers generates Pythagorean triples. But does this squaring method generate *all* the Pythagorean triples? Explain.

Tough Stuff.

27. Prove that at least one number in every Pythagorean triple must be even. Then prove that *exactly one* number in every primitive Pythagorean triple must be even.

28. Build an identity for generating primitive Pythagorean triples where the hypotenuse is a perfect square.

29. Hey, it's another division problem where we've provided one of the digits, an 8. The other digits are unknown.

```
                  X X X X X
       X X X ) X X X X X X X X
                 X X X
                 -----
                   X X X X
                     X X X
                   -------
                       X X X X
                       8 X X X
                       -------
```

30. Sketch the graph of the equation $x^y = y^x$ in the first quadrant. What point is that, and why?

Whaddaya mean, *what point*? THAT one. Duh.

9 *Stupefy-NING*

PROBLEM

Download this file for today's sketch:

`http://tinyurl.com/complexgsp`

Let v be a complex number with magnitude 2.

- Draw a shape to indicate where v could lie in the complex plane.
- Pick a value of v and square it: where does it go? Describe all the possible places where v^2 could lie.

Accio Sketch!

Magnitude 2, like Fred and George Weasley? Oh.

Sadly the Sorting Hat is on holiday. He'll be back Monday.

Important Stuff.

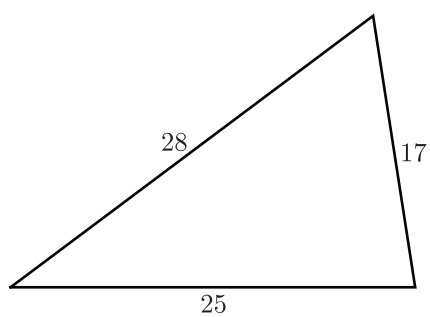
1. Use complex numbers to make three ridiculous primitive Pythagorean triples.

Riddikulus!

2. Find the area of this triangle. Keep careful track of your steps and do not use any "instant winner" formulas.

Expecto Heronus!

While it is possible to "eyeball" the triangle by picking just the right altitude, please additionally work through the calculations. They'll be helpful on the next page.



3. Explain why the area of a triangle is given by

$$A = \frac{1}{2}Pr$$

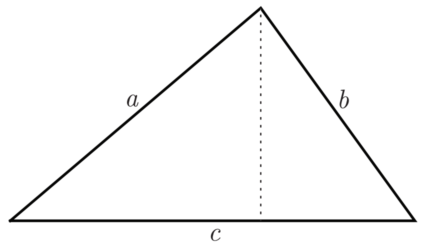
where P is the triangle's perimeter and r is the radius of its incircle.

4. Find the incircle radius of the triangle up there.

Wingardium Leviosa!

5. Find the area of this triangle by following the same steps you followed in problem 2.

Y U NO GIVE ME SIDE LENGTHS!!!!!



6. Find the incircle radius of a triangle whose side lengths are a, b, and c.

Specialis Revelio!

7. Take an 8-15-17 right triangle and a 5-12-13 right triangle and slap them together in some way to form a triangle with integer side lengths and integer area. You may need to scale one or both triangles first.

Triangles with integer side lengths and integer area are called *Hermionian triangles*.

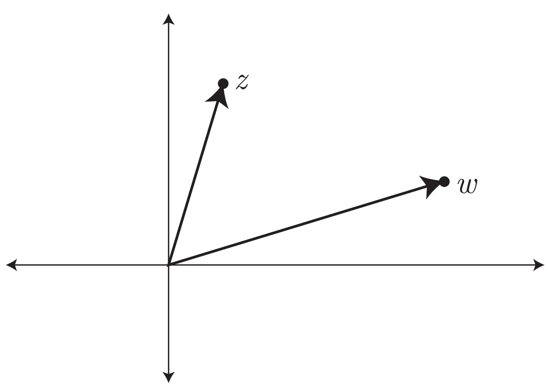
Engorgio!

8. Find at least two more ways to turn the 8-15-17 and the 5-12-13 into Hermionian triangles. Sweet, huh?

Ten points to Gryffindor!

9. Use your ridiculous Pythagorean triples to make a few ridiculous Hermionian triangles. Woo hoo!

10. The complex numbers z and w are plotted below. Plot $z + w$ and indicate how you know where it should be.



Neat Stuff.

11. Compute $(1 + 2i)(1 - 2i)$. Here, $1 - 2i$ is the conjugate of $1 + 2i$.

12. Find the magnitude and direction of $1 + 2i$.

Sketchpad can help you measure angles; remember, the "direction" angle is measured counterclockwise from the positive real axis.

13. Here's three complex numbers: $j = 3+i, m = 3+2i, i = i$. They can be connected to make a triangle.

(a) Plot these three numbers in the complex plane.

(b) Multiply each number by $1+2i$, then plot the results in the complex plane.

(c) Describe, as precisely as possible, how the new points' locations compare to the old.

(d) How does the area of the new triangle relate to the area of the old triangle?

Prior Incantato!

14. Find the surface area and volume of a 4 by 10 by 15 box and a 5 by 6 by 20 box.

15. Multiply these out until you don't feel like it anymore.

Impedimenta!

(a) $(x-4)(x-10)(x-15)$

(b) $(x-5)(x-6)(x-20)$

(c) $(4x-1)(10x-1)(15x-1)$

(d) $(x-\frac{1}{4})(x-\frac{1}{10})(x-\frac{1}{15})$

16. Find the dimensions of a box with the same total edge length and volume as the box with dimensions $\frac{1}{4}$ by $\frac{1}{10}$ by $\frac{1}{15}$.

17. So now you've got a cool formula for the area of a triangle given its sides. But those crazy formula people were busting out this crazy formula:

This thing s is called the *semiperimeter*.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Can you turn your cool formula into this one with an s?

18. Yesterday, Mary showed a sketch (available on the PCMI @ Mathforum website) where a circle of radius 2 is built, then a triangle is built using three points of tangency.

The file is called "mary's-cool-triangle.gsp". Blame Darryl! But it is cool.

(a) Why was this so cool?

(b) Investigate: is it possible to construct triangles with specific characteristics using this concept?

19. Find a different Heronian triangle with the same perimeter and area as the 17-25-28 triangle.

Geminio!

20. A triangle has side lengths 15, 7, and x, and its area and perimeter have the same numerical value. Find all possible values of x.

21. Find an equation for all the points that are α times as far from $(-1, 4)$ as from the line $y = 1$. (Let α be a positive number.) Describe how the shape created depends on the value of α.

Make sure what you find agrees with what you learned through Problems 12–14 from Day 8.

22. Suppose z is a complex number with magnitude 1 and direction θ. Then $z = a + bi$ with $a = \cos\theta$ and $b = \sin\theta$.
(a) Calculate z^2 directly by squaring $z = a + bi$. Wow!!
(b) Find a formula for $\cos 3\theta$.

If you're not working with trigonometry regularly, these last four will be relatively boring problems and are easily skipped.

23. Suppose z and w have magnitude 1, z has direction α and w has direction β. Let $z = a + bi$ and $w = c + di$.
(a) Calculate zw.
(b) What are the magnitude and direction of zw?
(c) What's the formula for $\sin(\alpha + \beta)$?

24. If $\tan A = a$ and $\tan B = b$, find a *simple* way to compute $\tan(A + B)$ without formulas.

Multiply! Find the right complex numbers and you're off to the races.

25. Pick angles A, B, and C that form a triangle. (You know what we mean.) Now let $\tan A = a, \tan B = b, \tan C = c$. Which is bigger, the product abc or the sum $a+b+c$? Try another triangle and compare. WHAT!

Petrificus Totalus!

Useless Stuff.

26. Solve for X.

Check my time, it's X.

$$\begin{aligned} Y_{est} &= X - 1 \\ 2D &= X \\ 2M_0 &= X + 1 \\ X + 2 &= \text{afterwards} \end{aligned}$$

Tough Stuff.

27. A triangle is uniquely determined by its side lengths, so it makes sense there is a formula for the area in terms of the three side lengths. Did you know that a triangle is also uniquely determined by its median lengths? Uh-huh! That means there is a formula for the area in terms of the lengths of the three medians. Go find it and rejoice because it's awesome.

Confundo!

28. There is also a formula for the area of a triangle in terms of the lengths of its three altitudes. It is less awesome but still possible.

Crucio!

29. There is also a formula for the area in terms of the lengths of its three angle bisectors. Good luck with that.

30. Prove that every Pythagorean triple must have a multiple of 3, a multiple of 4, and a multiple of 5 in it somewhere.

Looks like we've come to the end of the road. Still, I can't let go. I want a perfect body of problems. You may hate me but it ain't no lie. Next week it will be a whole new world... don't you dare close your eyes! It's 3 am, I must be lonely.

10 *New Directions*

PROBLEM

Download this file for today's sketch:
`http://tinyurl.com/complexgsp`

Let v be a complex number with magnitude 2.

- Draw where v could lie in the complex plane.
- Pick a value of v and cube it: where does it go? Describe all the possible places where v^3 could lie.
- Where does $v + i$ go and how does it move as v changes? What about $v + 3$? $2iv$? $v^2 + v$? $v^2 - v$?

If you "Plot Point," GSP will keep the point's coordinates when you translate the origin, zoom in, or zoom out. Use a plotted point to fix the radius of your circle to the coordinate plane. This helps if you need to change your scale. Plot points also advance the storyline!

You might be interested in experimenting with the "Trace" or "Locus" functions in GSP. Sue: "You might want to put us all out of our misery and shave off that Chia Pet."

Important Stuff.

1. Here are two Pythagorean triples: 5-12-13 and 85-132-157. Use them to produce more than one Heronian triangle. Then find its area and incircle radius using any method you like.

You can do it however you like, however you like. . .

2. Solve these quadratic equations.
(a) $(x - 3)(x - 17) = 0$
(b) $(2x - 3)(2x - 17) = 0$
(c) $(10x - 3)(10x - 17) = 0$
(d) $(100x - 3)(100x - 17) = 0$

FACT0R'D!!!!!

3. Solve these quadratic equations. We're not going to do your dirty work *this* time. Please, no formulas.
(a) $x^2 - 14x + 45 = 0$
(b) $4x^2 - 28x + 45 = 0$
(c) $100x^2 - 140x + 45 = 0$
(d) $10000x^2 - 1400x + 45 = 0$

Dude, use the first one to help you do the second one!! Dude!! Dude.

4. (a) Write a quadratic equation whose solutions are $x = \frac{2}{3}$ and $x = -\frac{4}{7}$.
(b) Oh, we meant one with no fractions, and completely multiplied out, and with 0 by itself on one side.
(c) Did you know that $\frac{2}{3} - \frac{4}{7} = \frac{2}{21}$? Interesting. Discuss.
(d) In the quadratic equation $ax^2 + bx + c = 0$, what is the sum of the two solutions? What is the *average* of the two solutions?

Sue: "You know what they say? Those who can't, teach. Turns out, maybe you actually can."

5. Let $f(x) = ax^2 + bx + c$, and use the quadratic you used in the previous problem.
(a) Graph $f(x)$ using your favorite technology.
(b) What are the x-intercepts of the graph?
(c) What is the x-coordinate of the vertex?

The pencil was invented in 1795. Papyrus was first manufactured by the Egyptians around 3000 BCE. Beat that.

The vertex is the turny spot.

6. What is the x-coordinate of the vertex of $f(x) = ax^2+bx+c$ in terms of a, b, and maybe c?

7. Consider points $P(x, 0)$, $Q(3, 0)$, $R(17, 0)$, $S(1, 0)$, and $T(11, 0)$.
(a) Define $a(P)$ to be the distance PQ. What does the graph of $a(P)$ look like, and where is its minimum?
(b) Redefine $a(P)$ to be the sum of the distances PQ and PR. What does the graph of $a(P)$ look like, and where is its minimum achieved?
(c) Redefine $a(P)$ to be the sum of the distances PQ, PR, and PS.
(d) Redefine $a(P)$ to be the sum of the distances PQ, PR, PS, and PT. What does the graph of $a(P)$ look like, and where is its minimum achieved?

Again, again! Hooray for repetition!

8. Consider points $P(x, 0)$, $Q(3, 0)$, $R(17, 0)$, $S(1, 0)$, and $T(11, 0)$.
(a) Define $b(P)$ to be the *square* of the distance PQ. What does the graph of $b(P)$ look like, and where is its minimum achieved?
(b) Redefine $b(P)$ to be the *sum of the squares* of the distances PQ and PR. What does the graph of $b(P)$ look like, and where is its minimum achieved?
(c) Redefine $b(P)$ to be the sum of the squares of the distances PQ, PR, and PS.
(d) Redefine $b(P)$ to be the sum of the squares of the distances PQ, PR, PS, and PT. What does the graph of $b(P)$ look like, and where is its minimum achieved?

Again, again! Hooray for repetition!

Wink wink, nudge nudge, know what I mean?

9. Which of the following points is a total of 30 units away from $(-9, 0)$ and $(9, 0)$?

(a) $(-15, 0)$	(c) $(6, 11)$	(e) $(12, b)$
(b) $(0, 12)$	(d) $(9, -\frac{48}{5})$	(f) (x, y)

10. The distance from (x, y) to the point $(9, 0)$ is $\frac{3}{5}$ of its distance to the line $x = 25$. Which of the following points makes that statement true?

(a) $(15, 0)$	(c) $(13, 6)$	(e) $(-12, b)$
(b) $(0, -12)$	(d) $(9, \frac{48}{5})$	(f) (x, y)

Neat Stuff.

11. The *eccentricity of an ellipse* is the ratio $\frac{c}{a}$ of two distances: the distance from the center of the ellipse to a focus (called c), and the distance from the center of the ellipse to a vertex (called a). For any ellipse discovered in Important Stuff, compute its eccentricity.

The eccentricity of Glee is Emma Pillsbury.

12. The distance from (x, y) to the point $(9, 0)$ is $\frac{4}{5}$ of its distance to the line $y = x$. Which of the following points makes that statement true?

(a) $(7, 2)$	(c) $(11, -14)$	(e) $(27, -18)$
(b) $(23, -2)$	(d) $(19, -19)$	(f) (x, y)

Don't stop believing that you can do this! Sue: "If I hear one song from that classic rock outfit Journey, I will start pulling catheters."

13. Show that this is true:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

Squares, squares, everywhere. Sue: "I will kick you square in the taco."

14. Suppose m can be written as the sum of two squares, and n can also be written as the sum of two squares. Prove that mn can *also* be written as the sum of two squares.

15. Use the result in problem 13 to prove that when two complex numbers are multiplied, their magnitudes are multiplied.

16. Suppose the sum and product of two complex numbers z and w are both real. Prove that either z and w are both real numbers, or $w = \overline{z}$.

17. Find the eccentricity of the shape in problem 12.

Maybe Rachel is the eccentricity. Actually, the real eccentricity is that the school has a Slurpee machine. Sweet!

18. Use multiplication of complex numbers to find a formula for $\tan(\alpha + \beta)$ in terms of $\tan \alpha$ and $\tan \beta$.

19. Find two values of $\tan \alpha$ and $\tan \beta$ so that $\tan(\alpha + \beta) = 5$.

20. Calculate this:

$$\frac{(5 + i)^4}{(239 + i)}$$

Any guesses on what this might be useful for?

Useful?! Math is supposed to be useful? Sue: "Asking someone to believe in a fantasy, no matter how comforting, is cruel."

Tough Stuff.

21. Let S be the set of complex numbers z with magnitude 2. Find a function (such as $z \mapsto z^2$) that produces an ellipse (and *not* a circle) as the output when S is used as the input.

Sue: "You are the honey X, nature's most ferocious animal. Look it up on Youtube."

22. The complex number w is a member of set M if the rule $z \mapsto z^2 + w$ with starting point 0 never has $|z| > 2$.

(a) Find some numbers w that are in set M, and some that aren't.

(b) Find a maximum bound for the *area* of the shape made by set M in the complex plane.

(c) What does set M look like if it is graphed in the complex plane?

We mean here that the rule is executed repeatedly. For example, if $w = i$ then the sequence of z values is $0, i, -1+i, -i \ldots$ and the hope is that none of those has magnitude more than 2.

23. What shapes tesselate in the hyperbolic plane?

24. (a) Evaluate $\tan 89°$ to two decimal places.

(b) How many degrees are in 1 radian? Give your answer to three decimal places.

(c) What is going on here? Is this a coincidence?

Coincidence? I think not!

11 *Rainbow Connection*

PROBLEM

Here is David's data.

x	y
1	1
3	8
5	3
7	8

Decide, without using anything other than your brain and perhaps some paper and pencil, on an equation for a line that "best" represents his data table. We ♢ an equation! Be prepared to defend your choice: why did you pick this line above others?

Kermit: "It's okay, Peter, you can be yourself." Peter Sellers: "There is no me. I do not exist. There used to be a me but I had it surgically removed."

Important Stuff.

1. One way to test the "badness" of a line toward data is to compute the *sum of squared errors*. Use the data from the box above.

POINTS... IN... SPAAAAAACE!

(a) Trevor used the line $y = x$ and wants you to confirm that the sum of squared errors is 30.

(b) Marakina used $y = x - 1$ and Olimpia used $y = x + 1$. How'd they do?

(c) Barb says there must be a rule, in terms of b, for the sum of squared errors of $y = x + b$. Go find it.

(d) Kate screams "OMG QUADRATIC" and finds the best possible b. Join her!

Movin' right along...

2. So, given a slope, it's possible to find one best line of that slope for the data (according to the sum of squared errors, anywho). Hop to it, and find the best line for each slope.

Divvy up the work however you like. However you like.

(a) slope 2
(b) slope 3
(c) slope 4
(d) slope 0
(e) slope -1

3. Graph the data and all six of the lines from the last two problems on the same axes. What!!

Wow, that is just gonzo. Or maybe more like Lew Zealand and the boomerang fish.

4. Marcelle hands you this new data set.

x	y
-3	-4
-1	3
1	-2
3	3

In terms of m, find the sum of squared errors of $y = mx$. Then determine the Best m Ever!

Fozzie once did a tribute to Marcel Marceau by impersonating a man standing on one leg, then a man standing on no legs. Waldorf said his act didn't have a leg to stand on. OH ho ho ho.

5. Type the original four points into a piece of technology and tell it to compute a linear regression.

(a) What is the slope of the linear regression?

(b) What important point does the graph of the linear regression pass through?

(c) Compute the badness of the linear regression. Is it simply the "best"?

Don't use Sketchpad, since it doesn't know how to compute linear regressions automatically.

6. A second way to test the "badness" of a line toward data is to compute the *sum of absolute errors*. Use the data from the box not above.

(a) Lauren used the line $y = x$ and wants you to confirm that the sum of absolute errors is 8.

(b) James used $y = x - 1$ and Timon used $y = x + 1$. How'd they do?

(c) Allison got fancy and used $y = x + \frac{1}{2}$. How'd she do?

(d) Becky says there must be a rule, in terms of b, for the sum of absolute errors of $y = x + b$. Go find it.

(e) Sketch the graph of Becky's rule as a function of b. What do you notice?

This is different from the *sum of absolut errors*, calculated by drinking vodka inappropriately. Wocka wocka!

We interrupt this problem for a Muppet News Flash: There is no news tonight.

Neat Stuff.

7. In yesterday's problem in the box, Shirley used Geometer's Sketchpad to visualize the locus of all points $v^2 + v$ when the complex number v has magnitude 2. Now, check out this awesome file:

`http://tinyurl.com/comppolygsp`

Explain, using the file, that $z = 2$ is a solution to the equation

$$z^2 + z = 6$$

Then find *all* of the solutions to $z^2 + z = 6$, resizing the circle as necessary to track them down.

Sorry, we meant *Randall* here, but he keeps telling us to call him Shirley.

This sketch is barely-controlled electric mayhem. Gonzo: "It's a question of mind over matter." Waldorf: "Well, we don't mind, and you don't matter! OH ho ho ho!"

8. Use the file to find at least one solution to each of these equations.
(a) $z^2 - z = 6$
(b) $z^2 - z = 12$
(c) $z^2 - z = -9$ (do a quick approximation)
(d) $z^2 - z = 20$

Start by using the sliders to change the function, then set the green circle's radius to 3.

"We're going to need a bigger circle."

9. Take a number z in the complex plane. How can you construct the conjugate $\overline{z}$? What's the conjugate of $z^2 + z$?

Take a number? Is this a deli?

10. Zee deestunce-a frum (x, y) tu zee pueent $(25, 0)$ is 30 mure-a thun its deestunce-a tu zee pueent $(-25, 0)$. Vhet pueents (x, y) meke-a thet stetement trooe-a? Bork, bork, bork!

Iff yuoor unsver gues thruoogh $(15, 0)$, yuoo ere-a oofer cuukeeng sumetheeng.

11. The distance from (x, y) to the point $(25, 0)$ is $\frac{5}{3}$ of its distance to the line $x = 9$. Test some points, then find an equation!

12. The *eccentricity of a hyperbola* is the ratio $\frac{c}{a}$ of two distances: the distance from the center of the hyperbola to a focus (called c), and the distance from the center of the hyperbola to a vertex (called a). For any hyperbola discovered in Important Stuff, compute its eccentricity.

The eccentricity of the Muppets is Sam the Eagle, since everyone else is pretty much a wacko.

13. For the data in the box problem find the unique line with the smallest sum of absolute errors.

14. Brian hands you this list of points:

t	h
1	3
1	7
3	5
5	3
5	7

Find the line that has the smallest sum of absolute errors. But wait, there's more! Find them all.

This data is 5 observations: t is time in seconds, and h is the number of times Animal banged his head against a cymbal in that many seconds.

15. Let $z = 34 + i$ and $w = 55 + i$. The product zw can be written as a scalar multiple of another complex number in the form (something) $+\, i$. What's the something? (Oh, what about $\frac{z}{w}$?)

16. Let $a = 2 + i$, $b = 5 + i$, $c = 13 + i$, $d = 21 + i$. Find the direction of the product $abcd$. Can this be generalized?

17. Find the three solutions to $x^3 = i$ using Sketchpad or some other means.

34 and 55, you say? I'd be fibbing if I said I'd never seen *those* numbers before. Perhaps the answer is 89. This problem is obviously inspired by that bunny episode of Veterinarian's Hospital. . . the continuing story of a quack who's gone to the dogs. It's a real rabbit and Costello routine.

18. (a) Multiply out $(a + bi)^3$.
(b) If z is a complex number with magnitude 1 and direction θ, what are its coordinates?
(c) Write rules for $\cos 3\theta$ and $\sin 3\theta$ based on the first two parts of this problem.
(d) The rule for $\cos 3\theta$ is $4\cos^3\theta - 3\cos\theta$. What up with that?

If you're not a trig fan, just move along, nothing to see here.

19. (a) Expand the expression $(x + 2)^3$.
(b) How many faces are there on a cube? How many edges? How many vertices? Wacky.
(c) Does this pattern continue at all, in lower or higher dimensions?

What the hexahedron is going on here?

20. Suppose $\tan A = \frac{1}{x}$ and $\tan B = \frac{1}{y}$, where x and y are integers. Is it possible for $\tan(A + B)$ to also be in the form $\frac{1}{z}$ for some other integer z? If so, how?

21. Let z be a complex number with magnitude 1. If $z = x+yi$ and $y > 0$, explain why $y = \sqrt{1 - x^2}$.

22. Let $z = x + (\sqrt{1 - x^2})i$ as in the last problem.
(a) Expand z^2 and look at the real part. Zeros?
(b) Expand z^3 and look at the real part. Zeros? Connections?
(c) Expand z^4 and look at the real part. Hmmmmm!
(d) Graph the real part of z^2, z^3, z^4 as a function of x.
(e) What might z^n look like?

The TI-Nspire can handle this, believe it or not.

Pafnuty says hi. Who's that? Look it up!

Tough Stuff.

23. So, you managed to find two noncongruent boxes with the same surface area and volume? Good, but can you find *three* noncongruent boxes with the same surface area and volume? What now!!

"Hey, these problems aren't half bad." "Nope, they're *all* bad! OH ho ho ho!"

24. How does the Law of Cosines translate into the hyperbolic plane?

25. Let $f_n(x) = \cos x \cos 2x \dots \cos nx$. For which integers $1 \leq n \leq 10$ is the integral from 0 to 2π of $f_n(x)$ non-zero?

26. Meep. Meep. Meep meep. Meep meep meep. Meep meep meep meep meep. Meep meep meep meep meep meep meep meep?

Just when you think this class is terrible something wonderful happens. . . it ends! OH ho ho ho!

12 *5-Minute Longs*

PROBLEM

Sarah has this set of data giving n, the number of people enjoying the problem set on day d.

d	n
2	40
3	30
5	35
6	20
9	10

Chris really wants to be Sarah's new BFF, so he wants to find the BFF (Best Fit Forever) line for this data. Without fancy technology, use what you learned yesterday to help Chris find the BFF line that minimizes the sum of squared errors. What "badness" did you get for this line? Check to see that other "close" lines have higher badness.

Determine the BFF line's prediction for the number of people enjoying today's problem set.

Sweet, it's non-decreasing! We've done our job.

It's 106 miles to Chicago, we got a full tank of gas, half a pack of cigarettes, it's dark, and we're wearing sunglasses. Hit it.

Important Stuff.

1. (a) Write an expression for the "badness" (sum of squared errors) for an arbitrary line $y = mx + b$ for the data above. *Don't expand any squared things in parentheses!* Make sure everyone at your table agrees on an answer.

   (b) The crazy expression you made is equivalent to this other crazy one:

$$5(b + 5m - 27)^2 + 30(m + 4)^2 + 100$$

   So what? What's this new expression got going for it?

Check it if you want to. I don't recommend it!

2. Use the badness expression $5(b+5m-27)^2+30(m+4)^2+100$ to find one or two lines whose badness is exactly 130.

3. If A, B, and C are positive integers with $A \leq B \leq C$, find all possible solutions to

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = \frac{1}{2}$$

What is the largest possible value of A? Given A, what is the largest possible value of B? Keep at this until you exhaust all options.

4. Multiply out the equation in problem 3 so there are no fractions. What do you notice?

5. Find all possible rectangular boxes with integer side lengths whose surface areas are numerically equal to their volumes.

Good morning. In less than an hour, aircraft from here will join others from around the world. And you will be launching the largest aerial battle in this history of mankind. Mankind – that word should have new meaning for all of us today. We can't be consumed by our petty differences anymore. We will be united in our common interests. Perhaps its fate that today is the 4th of July, and you will once again be fighting for our freedom, not from tyranny, oppression, or persecution – but from annihilation. We're fighting for our right to live, to exist. And should we win the day, the 4th of July will no longer be known as an American holiday, but as the day when the world declared in one voice: We will not go quietly into the night! We will not vanish without a fight! We're going to live on! We're going to survive! Today, we celebrate our Independence Day!

Neat Stuff.

6. The distance from (x, y) to the point $(1, 0)$ is α times its distance to the line $x = -1$. Figure out how the graph of this set of points depends on the number α.

7. The function $f(z) = z^3+4z$ is an *odd* function. What does an odd function do to the set of all complex numbers z with magnitude 2? Use this awesome Geometer's Sketchpad file to investigate:

`http://tinyurl.com/comppolygsp`

While you're at it, what about even functions?

8. Find or approximate the *three different* solutions to the equation

$$z^3 + z + 2 = 0$$

9. Find the *five* complex numbers that make $z^5 = 1$. Don't do this by factoring, use Sketchpad! While you're at it, find the five complex numbers that make $z^5 = -1$.

10. Find all *ten* solutions to $z^{10} = 1$, and plot them in the complex plane.

If $z^{10} = 1$, what are the possible values of z^5?

11. Consider the function $w = f(z) = z^2 + 4z + 5$.

(a) Mary uses a really small circle S as input. What does the output look like? By small we mean small. No, smaller!

(b) Tina likes large circles so she grabs Mary's circle and drags it until it's huge. What happens? Does the output ever cross the origin? How many times?

12. The badness expression in problem 1 can also be written this way:

$$155\left(m + \frac{5}{31}b - \frac{111}{31}\right)^2 + \frac{30}{31}(b-47)^2 + 100$$

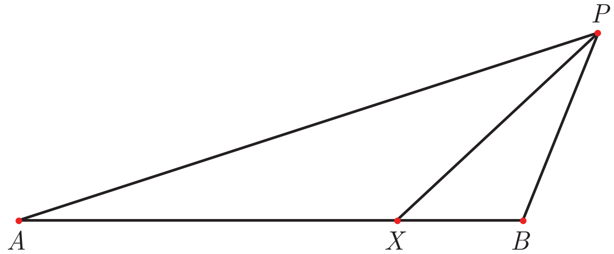
Explain what this is good for, and see if you can find any similar ways to write the expression.

13. Compute the "badness" function for the data from yesterday's problem in the box as a function of m and b. Then, rewrite that expression in a clever way to find the value of m and b that minimizes the badness.

Over? Did you say "over"? Nothing is over until we decide it is! Was it over when the Germans bombed Pearl Harbor? Hell no! And it ain't over now. 'Cause when the goin' gets tough . . . . . . the tough get goin'! Who's with me? Let's go! What happened to the Delta I used to know? Where's the spirit? Where's the guts, huh? "Ooh, we're afraid to go with you Bluto, we might get in trouble." Not me! I'm not gonna take this. We're just the guys to do it. LET'S DO IT!!

Let A and B be two points. What is the locus of points P such that the ratio AP/PB is a fixed number r? Check out problems 14 through 17 if you're interested, otherwise skip 'em.

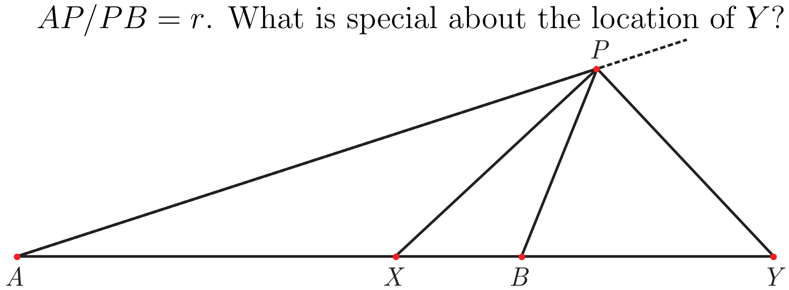
14. Let A and B be two points and let P be a point that is not collinear with A and B. There is a point X along the segment $\overline{AB}$ such that $AX/XB = AP/PB = r$. What is special about the location of X? Feel free to use Geometer's Sketchpad to investigate this. To make GSP construct point X at the proper spot, measure AP and PB, then calculate the ratio $AP/(AP + PB)$. Double-click point A to mark it as the the center of a dilation, then dilate the point B using the marked ratio $AP/(AP + PB)$.



15. Prove what you found in problem 14 by determining the ratio of the areas of triangles AXP and BXP in two different ways.

16. There's another point on the ray $\overrightarrow{AB}$ that is important. Let Y be a point on $\overrightarrow{AB}$ (besides X) for which $AY/YB = AP/PB = r$. What is special about the location of Y?

That dotted line is not a mistake. . .



17. Determine the measure of $\angle XPY$ and its consequences for the possible locations of point P.

Do not taunt Happy Fun Ball.

18. In radians, find the smallest positive x solving $\tan x = \frac{1}{5}$.

19. Let $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. Values of $f(x)$ converge when x is between -1 and 1.

It's a Taylor series! *Run away!* Oh, sorry, it's only a Maclaurin series. False alarm.

(a) Approximate $f(\frac{1}{5})$ to six decimal places. You might need to take a few terms.

(b) Approximate $f(\frac{1}{239})$ to six decimal places. Did this take more terms, or less terms?

(c) Evaluate $16f(\frac{1}{5}) - 4f(\frac{1}{239})$ to five decimal places. What!

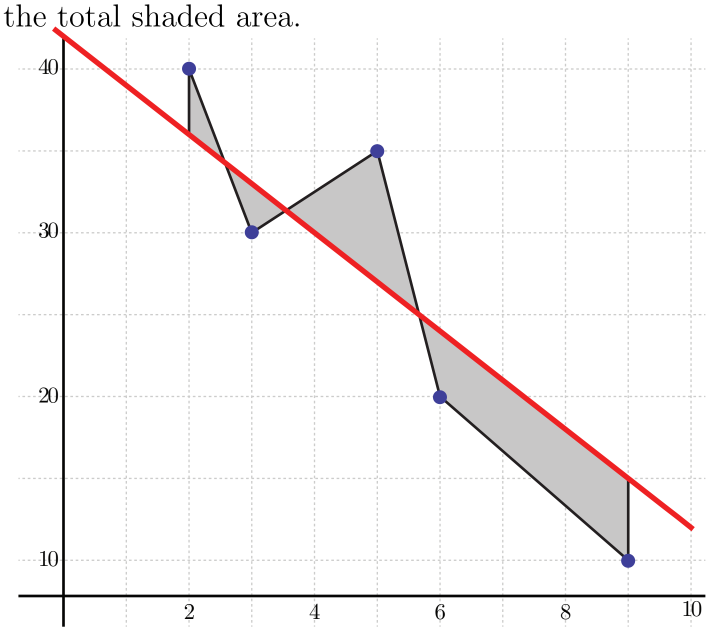
(d) Hey, we forgot: find the direction of the complex number

$$\frac{(5+i)^4}{239+i}$$

What might this be useful for??

Tough Stuff.

20. Here's a graph of a candidate fit line and the data from today's problem in the box. Find the line that minimizes the total shaded area.



21. Factoring $x^5 - 1$ over the integers isn't too bad, it's

$$(x-1)(x^4 + x^3 + x^2 + x + 1)$$

The tough part is to factor that quartic piece into two quadratic factors, where each quadratic factor has real coefficients (not restricted to integers or fractions). Find the factoring, then look for connections between the factoring and the construction of a regular pentagon.

13 *Dining with the STaRs*

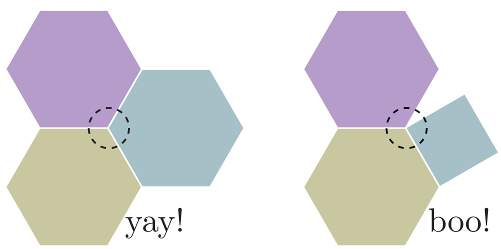
PROBLEM

Karen puts 3 regular polygons together so they meet at a point with no overlap and no empty space. One option is to shove together 3 hexagons, but there are other ways.

Don't worry about whether or not these polygons will fill the entire plane this way. We're just looking at one vertex.

Find all the ways you can fit 3 regular polygons together at a point. Write each set in increasing order by the number of sides in the polygons.

Three. Three regular polygons. Ah ha ha ha.



WHAT!!! WHY!!!

Important Stuff.

1. Find all the ways you can fit 4 regular polygons together at a point.

Four! Four regular polygons, ah ha ha ha.

1. Find all the positive integer solutions to this awesome equation, where $G \le A \le B \le E$:

$$\frac{1}{G} + \frac{1}{A} + \frac{1}{B} + \frac{1}{E} = 1$$

3. Find all the ways you can fit 5 regular polygons together at a point... 6 regular polygons... 7 regular polygons.

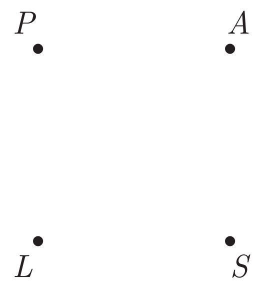
Five!! Five regular polygons, ah ha ha ha!

4. Draw a picture of what it looks like when you do math.

Or, a picture of Marie Osmond fainting. Your choice.

5. Pedro, Arden, Lindsey, and Sandy form a square. They just met at PCMI and now they want to connect themselves. Connect them with paths so the total distance of all the paths is as small as possible. (Two people will be considered *connected* if there is any path from one to the other. Directly connecting is okay, but not required.)

The four people are friends so they will be *PALS*. When they go to the pool, they do *LAPS*. When in Europe, they visit the *ALPS*. If they get in a fight, they *SLAP*. And when Therese joins them for a food fight, they *SPLAT*.



Neat Stuff.

6. Find some right cylinders that have the same numerical value for volume and surface area.

The surface area of these right cylinders includes the top and bottom discs.

7. If Jocelyn tells you a right cylinder's total surface area and total volume, is that enough information to uniquely determine its dimensions? Try it!

These cylinders are just a series of tubes! Not many people know this, but I owned the first radio in Springfield. Not much on the air then, just Edison reciting the alphabet over and over. A, he'd say; then B. C would usually follow.

8. Go back to yesterday's data set and look for a line with the lowest sum of absolute errors. In the m-b plane, where m is the slope and b is the n-intercept, what is the locus of all points (m, b) corresponding to lines with the same "badness"?

9. Open this awesome Geometer's Sketchpad file:

`http://tinyurl.com/comppolygsp`

Investigate how the function $f(z) = z^2 - 2z + 4$ operates on different "magnitude circles" (circles centered at 0). Estimate with very poor accuracy the two solutions to $z^2 - 2z + 4 = 0$. (They are both on the same magnitude circle.)

10. You have 30 seconds. Given $f(x) = x^3 + 6x + 3$, estimate $f(\frac{1}{100})$ and $f(100)$. GO.

29. . . 28. . . 27. . .

11. Consider the function $f(z) = z^3 + 6z + 3$.

(a) What does the output of $f(z)$ look like when you use a very small magnitude circle?

(b) What does the output of $f(z)$ look like when you use a very large magnitude circle?

(c) Explain how you know that there *must* be a magnitude circle that contains a point z with $f(z) = 0$.

This one has z instead of x. It's *like, totally* different!

If you can't tell, your circle isn't small enough yet.

12. You have 60 seconds. Given $f(x) = x^3 + 12x - 4$. Estimate each of the following. GO.

(a) $f(\frac{1}{100})$

(b) $f(-\frac{1}{100})$

(c) $f(100)$

(d) $f(-100)$

This one probably ended up on the cutting room floor of "Minute to Win It".

13. *Without using Sketchpad*, answer the same questions in problem 11 but for $f(z) = z^3 + 12z - 4$.

14. Convince yourself and everyone else that this is true:

> If $f(z)$ is a polynomial of degree n then there must be at least one complex number z with $f(z) = 0$ in the complex plane.

"Degree n" just means that the polynomial starts $f(z) = az^n + \cdots$

15. Use exterior angles to explain why the stuff with the regular polygons just happened.

16. So the tiling of the two hexagons and the square from the box didn't work out. *Or did it!*

(a) Calculate $\frac{1}{6} + \frac{1}{6} + \frac{1}{4}$. How much more than $\frac{1}{2}$ is it?

(b) How many "missing" degrees are there at the vertex? How many "missing" degrees would there be for 24 vertices?

(c) Is there a tiling with two hexagons and a square at each vertex? What does it look like?

As the Beastie Boys might say, "Another dimension another dimension another dimension another dimension".

17. There's another option: tilings that overlap. For example, a tiling of three heptagons (7-sided figures). What might that look like, and is there any relationship to the fractions?

18. Use Geometer's Sketchpad to construct the locus of points (x, y), whose distance to the point $(1, 0)$ is α times its distance to the line $x = -1$.

It's a plague of locusts! I haven't seen this many locusts since nineteen-dickety-two. We had to say dickety because the Kaiser had stolen our word "twenty".

19. Consider the function $f(z) = z^2 - z$.

(a) Find a solution to $z^2 - z = 1$ using Sketchpad.

(b) Find a way to use this sketch to build a golden rectangle.

20. Use Sketchpad to approximate the five solutions to $x^5 + 2x^2 + 10 = 0$. How many of the roots are real numbers?

If the last three digits of your raffle ticket precede 5309, you win! Claim yo prize.

21. Use Sketchpad to approximate the four solutions to $x^4 + 3x^2 + 1 = 0$. Yes, there are four solutions and not two. What happens, in general, to even functions? (Tougher: Find the four roots by factoring.)

22. Track down the solutions to $x^3 - 7x^2 + 15x - 9 = 0$. What does a "double root" look like in the complex plane? How many solutions does this equation have?

Poor Karen Carpenter. . . Ask Cal to give the punchline.

Review Your Stuff.

Historically, the final day is considered review. Because this is a self-reflective process of discovery, we think that an end product of this discovery might be some summarizing questions of what you might find valuable in this course. We would like these to get at what you think are important mathematical themes in the course, and also themes that might apply to what you teach. We hope this will be a valuable journey, but mostly we just want you to write two problems on any topics that have cropped up in the course. We may or may not use your review questions, depending on how much other material we write and on the color of the paper on which you submit your questions.

Is there really such a thing as a "self-reflective process of discovery"? Yes, there really is! Don't believe us? Ask Google. And don't forget to put quotes around that.

Tough Stuff.

23. Construct a regular pentagon using the quadratic factoring of $x^5 - 1$, then look for quadratic factorings of other polynomials in the form $x^n - 1$.

24. Find some connections between what we've been doing with tangent and Taylor series.

25. Use the Taylor series for $\sin x$, $\cos x$, and e^x to show that $e^{i\pi} + 1 = 0$. Woo!

26. What is one possible value of i^i? (Use a TI-Nspire in radian mode.) How on earth would one arrive at such a thing? Why did we say "one possible value"?

27. Solve this problem then claim your prize:
`http://www.claymath.org/millennium/Riemann_Hypothesis/`
Don't forget to give us 50% of your prize.

We can't bust heads like we used to, but we have our ways. One trick is to tell 'em stories that don't go anywhere, like the time I caught the ferry over to Shelbyville. I needed a new heel for my shoe, so I decided to go to Morganville, which is what they called Shelbyville in those days. So I tied an onion to my belt, which was the style at the time. Now, to take the ferry cost a nickel, and in those days, nickels had pictures of bumblebees on 'em. "Give me five bees for a quarter", you'd say. Now where were we? Oh yeah: the important thing was I had an onion on my belt, which was the style at the time. They didn't have white onions because of the war. The only thing you could get was those big yellow ones.

14 *You + Me = Us*

PROBLEM

Open this awesome Geometer's Sketchpad file:
`http://tinyurl.com/comppolygsp`

Let's investigate the function $w = f(z) = z^3 - 2z^2 - 5z + 6$.

- When we restrict the input z to a very small magnitude circle, what does the output look like?
- When we restrict the input z to a very large magnitude circle, what does the output look like?
- Explain how you know that there *must* be a magnitude circle that contains a point z with $f(z) = 0$.

I know my calculus... it says you plus me equals ganas...

Come on, let's!

Embiggen your circle! Biggen!

Amazingly Fantastic Stuff.

1. David says he can use the method of the box problem to show that, for *any* polynomial, there is at least one z with $f(z) = 0$. He's just nuts, right?

Your Stuff.

2. A tetrahedron has three of its faces meeting together at right angles. Find a relationship between the areas of its four faces.

It's like a corner of a cube sliced out. Take that, Pythagoras!

3. Judi, Dan, Cuong and Katherine are walking to 7-11 located at point $(7, 11)$ at 7:11 p.m. (on July 11th for free Slurpees, of course). Dan bet Katherine she couldn't find the area of the triangle created by our trip from Prospector $(86, 2)$ to 7-11 at $(7, 11)$ and back to her room $(7, -7)$. Did she collect? What is the area?

Free Slurpees may be available on other days, but only in rare circumstances.

4. Faynna, Joey, and Danielle are standing at the vertices of triangle FJD. At what point A should Anna stand to minimize $FA+JA+DA$? Let L be this minimum distance. Define Anna's badness, $B(A)$, to be $FA+JA+DA$. What does the locus of points of a fixed badness look like? What if Anna stands outside of triangle FJD? What is the locus of points with a fixed $B(A)^2$?

Clearly, Anna should stand at the nearest Chili's for FAJADAs. And, no, Anna is not actually bad, except possibly in the Michael Jackson or Domenico Modugno sense.

5. How can you prove that the 120-degree point has the shortest total distance to the three vertices in a triangle? Does your proof work for *any* triangle?

6. Find two non-congruent triangles with integer side lengths that share the same area, A, and the same perimeter, P.

7. Start with $z = 1 + i$ and graph (sketch) several positive integer powers of it. Then do the same thing, but starting with $w = \frac{1}{2} + \frac{1}{2}i$. What is the value of the infinite sum $1 + w + w^2 + w^3 + \cdots$?

8. Find all positive integer solutions to the equation below, where $A \le B \le C$.

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = \frac{2}{3}$$

If you find a connection between this problem and others from these sessions, let us know, because we looked!

9. A more general version of the unit fraction problem is

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \frac{k}{2}$$

where n, k, and all x_i are positive integers. Find some conditions on n and k for which this problem has a solution.

10. Go back to problem 5 from Day 13, but now suppose that Mike, Mariam, Matthew and Monty are arranged in a rectangle rather than a square. How does the solution change? If they are arranged in a general quadrilateral, will the solution be unique? What if there are five points in an arbitrary arrangement?

The four people are friends so they will be *MMMM*. When they have a satisfying meal, they go *MMMM*. Their favorite Crash Test Dummies song is *MMMM MMMM MMMM*. If they don't know the words at karaoke, they *MMMM*. And when Heron joins them to make a pentagon, they go *MMHMM*.

11. Three regular polygons can meet at a point with no overlap and no empty space in several ways. But which ones tessellate the plane?

12. For each regular -gon, find the side length that will make its area numerically equal to its perimeter.

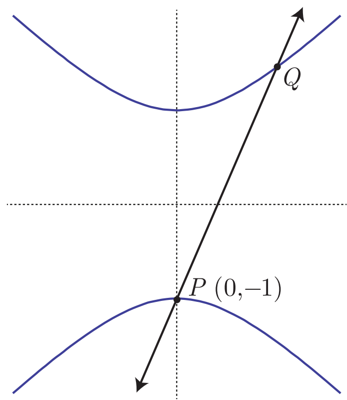
(a) 3-gon
(b) 4-gon
(c) 6-gon
(d) 8-gon
(e) 12-gon
(f) n-gon

13. Given a specific surface area and volume, how can we find all the boxes with that surface area and volume? What if we are restricted to boxes with integer side lengths?

14. Can every Heronian triangle be deconstructed into two Pythagorean triples?

15. On Day 4 we intersected the unit circle $y^2 + x^2 = 1$ with a line of slope m passing through $(0, -1)$. The other intersection with the unit circle could be found in terms of m, and magical things happened: we could find as many rational-coordinate points on the unit circle as we wanted! Repeat this process with the unit *hyperbola* $y^2 - x^2 = 1$ with a line of slope m passing through $(0, -1)$. Find the other intersection in terms of m and enjoy the magic.

It's *magic*! If you squint hard enough, the hyperbola transforms into a unicorn.



16. Compute each value. Find all possible answers!

(a) i^2
(b) i^3
(c) $\sqrt{i}$
(d) $\sqrt[6]{i}$
(e) $(3 + 4i)^2$
(f) $\sqrt{3 + 4i}$
(g) $\sqrt{3 - 4i}$

DOES NOT COMPUTE. Meep meep meep.

17. Find another complex number whose direction is 60 degrees counterclockwise from these given complex numbers, as seen from the origin.

(a) $1 + i$
(b) $3 - 2i$
(c) 5
(d) $a + bi$

What are the roots of that thing with two numbers that add to whatever that thing is? Eh, you know what we mean.

18. Find a third point in the coordinate plane that will complete an equilateral triangle with these given two points.

(a) $(3, 4)$ and $(4, 5)$
(b) $(2, 1)$ and $(5, -1)$
(c) $(3x, 2)$ and $(3x + 5, 2)$
(d) (x, y) and (w, z)

Psst: the last problem might help you! Or it might not! OH ho ho ho.

19. The "taxicab distance" (a.k.a. "Manhattan distance" or "rectilinear distance") between any two points in a plane is equal to the sum of the absolute differences of their coordinates. If you think of the lattice points on a graph as being the intersection of streets and avenues, the taxicab distance between two points represents how far a taxi must go to get from one intersection to another *travelling only on the streets and avenues* of the plane. The taxicab distance between $(3, 5)$ and $(7, 3)$ is calculated as $|7 - 3| + |5 - 3| = 4 + 2 = 6$. Note that there are several paths between the two points that have this minimum path length.

In Park City, the taxicab distance is usually greater than the walking distance, and may even be slower at night.

(a) Jason is a fireman in brand new Gridville, which only has three houses at $(1, 4)$, $(6, 1)$, and $(9, 5)$. Where should his firehouse be built to minimize the average taxicab distance to the three houses? This Geometer's Sketchpad file may be useful:

`http://www.tinyurl.com/taxicabgsp`

Jason wanted to grow up to be a bow-tie-wearing lounge singer, or perhaps a night sky photographer, but both of these options fell through. So he's stuck as a fireman.

(b) Two more houses are built in Gridville at locations $(2, 7)$ and $(3, 6)$. Fortunately, Jason's firehouse is easily moved. Where should he move it?

(c) Stop and reflect on where these "best" points are and how they relate to the locations of the houses. Notice anything?

(d) A sixth house is built at $(6, 6)$. Find a new firehouse point for these six houses. And more! How many are there? Did you find six of them? Wait... why not?!

We aren't sure if Jason is telling the truth here, or if his pants are en Fuego.

(e) Farmville crops up nearby with two neighborhoods of houses. The houses are located at $(2, 8)$, $(2, 9)$, $(3, 7)$ and $(3, 10)$, as well as at $(5, 1)$, $(7, 4)$, $(8, 2)$, and $(10, 5)$. Find all the firehouse spots for this set of houses. Good luck and good night.

Or, in Jason's case, good morning! Be sure to wake him up!

20. Go back through the previous problem and find firehouse spots that have the lowest possible *maximum* taxicab distance to any house. How are these points related to the median?

21. There may or may not exist in the real world a *unicorn tetrahedron*. These are special tetrahedra defined by having integer edge lengths, integer areas of faces, and integer volumes. If they exist, and you construct one out of paper, you will see a magic unicorn—sparkling with glitter. Please, please, for all that is good in this world, show that in our world, a *unicorn tetrahedron* exists. Or be cruel and dash Bill's hopes and show that they don't.

Man, what were we smoking when we wrote this problem? Oh, right, *one of you* wrote it.

Which Bill are we talking about? There is Bill and then there is B̈ill who spells his name with an umlaut.

22. Is it possible for a clock to have 120 degrees between all three hands if it is
(a) a continuously-moving second hand?
(b) a clock with discrete second "ticks" (with minute and hour hands moving incrementally)?

23. Is it possible to spoken-word sing both "Goodnight Irene" and Jimmy Eat World's "The Middle" in the span of two minutes?

Gertrude would know the answer to this.

Our Other Stuff.

24. Yesterday's arm-waving guy with the accent and the soap said that for a regular octagon, the shortest path is made by using the boundary instead of creating interior "Steiner points". Do you see any particular reason why this would be?

25. Go back to Brian's data from problem 14 on Day 11. Suppose $h = mt + b$ is an attempt at a best fit line and its badness is measured using the sum of absolute errors. In the m-b plane, what is the locus of all points (m, b) corresponding the same badness?

26. Jemal asks you to close your eyes and imagine a four-dimensional hyper-box with dimensions A, B, C, and D. (He's really good at this. Ask him how.) Find all possible dimensions for this hyper-box so that $A \leq B \leq C \leq D$ are integers, and its four-dimensional hypersupervolumey-thing has the same numerical value as its total "face-volumes." Oof, my head hurts.

Jemal also asks you to imagine his driver's license expires in 2012, but we know that's just fiction.

27. When's the next time we'll see you again? Thanks for playing and thanks for teaching.