Park City Mathematics Institute

Geometrical Concepts from Constructions, Models, and Investigations

Rhombic Dodecahedron – Hidden Within or Surrounding the Cube?

By Joyce Frost and Kris Koch

Concepts: Geometric Solids, Measurement

Skills: Making constructions in three-dimensions (of cubes, square pyramids, and rhombic dodecahedrons), dissecting solids, finding lengths, recognizing three-dimensional relationships, problem solving

Grade Level/Strands: Secondary geometry

Class Time: 1-2 class periods to build the models, complete the worksheet, and conduct the classroom discussion.

Materials:

• Student Worksheet

• Teacher classroom set of Zome™ tools (if available)

If Zome™ tools are not available, each group of 4 students could use 96 clear standard sized drinking straws (24 for each person) and blue and yellow curling ribbon to build a total of 12 square pyramids per group.

Prerequisites:

Students should be familiar with the Pythagorean Theorem and have some knowledge of irrational numbers. For this student activity, assume in Figure 1 that the cube has edge length of 2 units, resulting in face diagonals of length $2\sqrt{2}$, and interior diagonals of $2\sqrt{3}$ units for the cube.

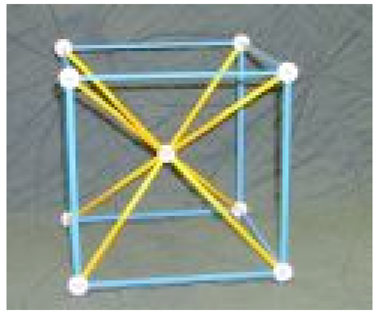


Figure 1: Cube with Interior Diagonals

The edge lengths for the square-based pyramids in Figure 2 are 2 units and $\sqrt{3}$ units.

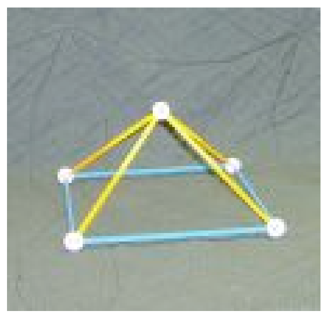
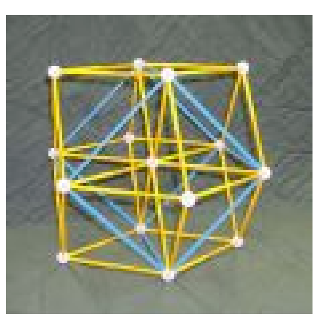


Figure 2: Square Pyramid Formed by Cube Face and Parts of Diagonals

Each of the square pyramids formed in Figure 1 are attached to either the original cube or a congruent cube forming the rhombic dodecahedron of Figure 3a. A different view with the original cube removed is seen in Figure 3b.



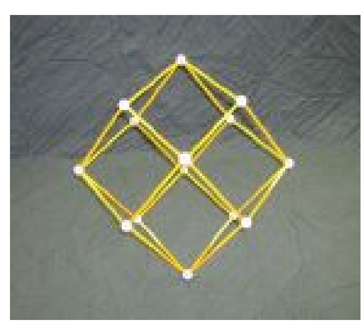


Figure 3: (a) Rhombic Dodecahedron (b) Rhombic Dodecahedron

The rhombic dodecahedron, dual of the cuboctahedron, is a fascinating polyhedron, although not one of the Platonic or Archimedian Solids. It is best known in nature as the crystalline shape of the garnet, the January birthstone. In this lesson, students explore the relationship of the rhombic dodecahedron to the cube and the space-filling or tessellating properties of both polyhedra. Students discover the relationship between the edges and diagonals of a cube. By analyzing the three different lengths found in a cube, students naturally derive the irrational numbers $\sqrt{1}, \sqrt{2}$, and $\sqrt{3}$. Additionally, they explore dissections of three-dimensional objects by comparing the unusual rhombic dodecahedron to the familiar cube. This is a mathematically rich problem to illustrate common algebraic and geometric concepts in two and three-dimensions.

Directions:

Prepare sample models of a cube (Figure 1), square pyramids (Figure 2), and rhombic dodecahedron (Figure 3a and 3b) using Zome tools For the cube, use 12 blue struts for edges and 8 balls for vertices. Complete with 8 yellow struts and 1 ball for the inside diagonals and center. Also prepare six square-based pyramids using 4 blue edges, 4 yellow diagonal edges, and 5 balls for vertices. The rhombic dodecahedron is built starting with a cube as in Figure 1 and is completed with square pyramids (Figure 2) placed on each of the six faces as in Figure 3a. Additionally, prepare a rhombic dodecahedron using 24 yellow struts and 14 white balls as in Figure 3b.

If using clear straws instead of Zome tools, prepare models using blue curling ribbon inside full length straws for the blue struts and yellow curling ribbon inside 7/8 length straws for yellow struts. For this activity, edge lengths are 2 units; thus yellow lengths would be $\sqrt{3}$ or may be approximated for this construction as 1.75 units. Therefore, a 1.75 unit is 7/8 of a 2–unit straw. Folding a piece of paper equal in length to a straw is an easy place to start. By folding this piece of paper in half three times to create eighths, it is easy to create a template to cut straws that are 7/8 the length of the original straws.

Arrange students in groups of 3 or 4. Each group needs a cup or Ziploc bag containing 12 blue struts, 8 yellow struts, and 9 white balls, and each student in the group needs a worksheet. If you have enough Zome tools, each group should also get several sets of 4 blue, 4 yellow, and 5 balls to make square-based pyramids to use as "jackets" to surround their cubes. Have students build their models and work through the worksheet. Circulate among the groups to answer questions and to check for understanding of key concepts. Make use of the prepared models to question students and probe student understanding. Allow ten to fifteen minutes at the end of class to bring the groups together as a large group and discuss findings.

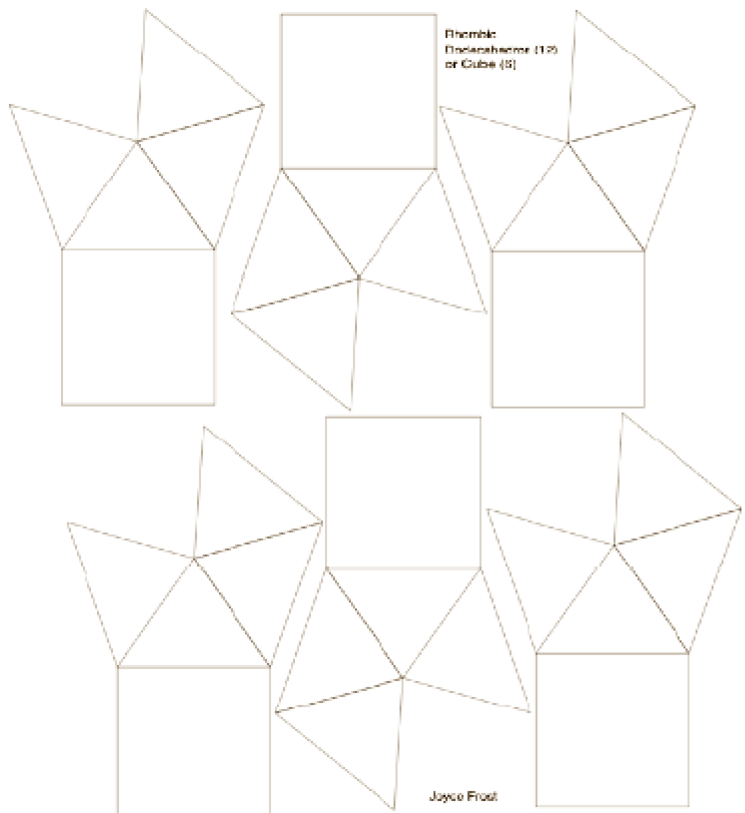
Questions that may be asked of students to help achieve the objectives of the lesson follow:

- How can you determine the lengths of the diagonals of a cube using the edge length?
- How are the edges and diagonals of the cube related?
- Why is the cube considered a space-tessellating, or space filling, shape?
- How can another space-tessellating shape, the rhombic dodecahedron, be created using the diagonals of the cube?
- How can one prove that the rhombic dodecahedron is a space-tessellating polyhedron by analyzing its relationship to the cube from which it was constructed?
- How can the volume of the cube and its corresponding rhombic dodecahedron be computed and compared?
- How can the name, rhombic dodecahedron, be explained based on the number and shape of its faces?
- How can one convincingly argue that, when attached to a cube, the two triangular faces of the square pyramids meet in the same plane and create a rhombus?

Extensions: More about Rhombic Dodecahedra

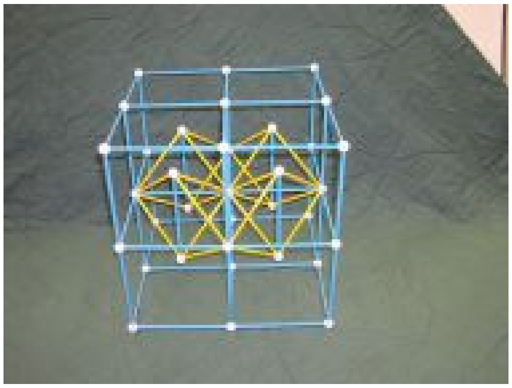
a. Search the Internet for information about rhombic dodecahedra.

b. Take the net below for the square pyramid pieces, print it on cardstock or glue the sheet to cardstock or _ of a manila file folder. Place the card stock/manila folder on contact paper, score the inside lines of each net, cut them out and tape into (convex) square pyramids. Attach six of these pyramids to one of the eleven hexomino configurations that form a cube. Work in pairs. One student can build the inside of a cube and the second student can create the outside jacket of the cube making a rhombic dodecahedron.



c. Using simple trigonometric functions, find the angles of the rhombi that create the faces of the rhombic dodecahedron.

d. A rotating ring of non-regular tetrahedra, sometimes called a "kaleidocycle," is a dissection of the rhombic dodecahedron. The kaleideocycle is shown in the interior of the figure below.



1. What's the ratio of the volume of the kaleidocycle to the rhombic dodecahedron?
2. How many tetrahedrons would make a complete rhombic dodecahedron?
3. Find the volume of one of those tetrahedra.

e. Search the Internet for Archimedian solids, Johnson solids, or Kepler solids as related topics.

f. List as many of the two-dimensional polygonal shapes that can be found when slicing through the rhombic dodecahedron with a plane as you can. Are any of the shapes regular?

g. Imagine the rhombic dodecahedron filling space and a plane cutting through it to create tessellations of the plane. Name possible figures that can be found that in the ensuing tessellations.

References

Cundy, H.M. and A.P. Rollett. *Mathematical Models: Third Edition*. Norfolk: Tarquin Publications, 1989.

Frost, J. and Cagle, P. *An Amazing Space Filling, Non-regular Tetrahedron*, http://mathforum.org/pcmi/hstp/resources/dodeca/paper.html

Holden, A. *Shapes, Space, and Symmetry*. New York: Columbia University Press, 1971.

Pierce, P. and S. *Polyhedra Primer*. Palo Alto: Dale Seymour Publications, 1978.

Schattschneider, D. and Walker, W. *M. C. Escher Kaleidocycles*, Petaluma, CA: Pomegranate Communications, 1987.

Additional Geometry Resources

Bulatov, V. *Polyhedra Collection,* http://www.physics.orst.edu/~bulatov/polyhedra/

Eppstein, David *The Geometry Junkyard: Three-dimensional Geometry,* http://www.ics.uci.edu/~eppstein/junkyard/3d.html

The Math Forum Internet Mathematics Library: Math Topics: Geometry,
http://mathforum.org/library/topics/geometry/

Answers to Worksheet:

1. 6 faces; 12 edges; 8 vertices

2. 2 diagonals; 12 face diagonals per cube?

3. 45-45-90 right triangle; all are congruent; the sides of a cube are congruent; the triangles angles are all isosceles right triangles; so the triangles are congruent by Side-Angle-Side.
4. The length of each face diagonal is $2\sqrt{2}$ units.

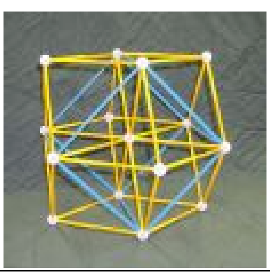
5. The diagonals of the cube are twice as long as a non-base edge of each pyramid.

6.

Length of edge of base	2 units
Length of altitude of pyramid	1 unit
Length of altitude of lateral face	$\sqrt{2}$ units
Length of lateral edge of pyramid	$\sqrt{3}$ units

7. 12 faces; the measures of the dihedral angles formed where three faces meet at a vertex on the new polyhedron are 120˚ (Three fit together inside the cube; all are congruent, and thus the measure of the angle formed must be 360˚/3.).

8. The dihedral angle of the cube is 90˚. The base dihedral angle of a square pyramid is 45˚ so the triangles meet where the surface is 45˚ + 90˚+ 45˚= 180˚ and must be coplanar.



9.

Length of edge	$\sqrt{3}$ units
Length of short diagonal	2 units
Length of long diagonal	$2\sqrt{2}$ units

10. The dihedral angles on the rhombi are 120˚so 3 make 360˚. A rhombic dodecahedron has twice the volume of the cube. In space, solid cubes alternate with the cubes filled with the 6 square pyramids.

11. 2 x 2 x 2 = 8 cubic units;

8/6 = 4/3 cubic units; by dividing the cube into 6 congruent pieces. The formula for the volume of a pyramid is one-third the base area times the height, or 1/3(2 x 2) = 4/3 cubic units. The rhombic dodecahedron has twice the volume of the cube or 16 cubic units. (Also, the rhombic dodecahedron is made up of 12 square-based pyramids. 12 x 4/3 = 16 cubic units.)

Answers to Extensions:

c. The angles of the rhombus are approximately 109.47° and 70.53° to the nearest hundredth. (Use basic trigonometric functions on the rhombus when it is separated into right triangles.)

d. 1. 4/12 = 1/3

2. 24

3. 16 x 1/24 = 2/3 cubic unit (16 is the volume of the Rhombidodecahedron built on a cube with sides of 2)

f. Student answers may vary, but among the answers you may find some of the following: square, trapezoid, equilateral, isosceles, and scalene triangles, octagon, heptagon, and hexagon. Squares and equilateral triangles are always regular. The octagon and hexagon can be regular or non-regular depending on the angle of the slice.

g. Student responses will vary, but some possible responses are: all hexagons, equilateral triangles, squares and hexagons, rhombi and hexagons, octagons and squares, and equilateral triangles and hexagons.