Session 11, July 23
Curve Fitting

1. Find a number x such that $x = 1 \bmod 11$ and $x = 0 \bmod 13$.
2. Find a number y such that $y = 0 \bmod 11$ and $y = 1 \bmod 13$.
3. Using Problems 1 and 2, find a number z such that $z = 5 \bmod 11$ and $z = 6 \bmod 13$.

The Chinese Remainder Theorem can be used to solve complicated modular arithmetic problems, so long as the mod's involved are relatively prime. By the way, if the mod's are *not* relatively prime, the general method of solution is to break up the problem into mod's that *are*, but we will have to leave that one.

4. Make sure you know how to use the Chinese Remainder Theorem to find the smallest positive integer x satisfying the following system of congruences. (If you are confident that you know how, skip this problem.)

$$\begin{aligned} x &= 5 \pmod{7} \\ x &= 3 \pmod{11} \\ x &= 11 \pmod{13} \end{aligned}$$

The Chinese Remainder Theorem will come up again really soon. Next though, we are going to look at several different methods of curve fitting. Curve fitting basically involves taking a table of values, and finding some function that works for the points in the table. The curve may be a linear function, a quadratic, any polynomial function ($Ax^n + Bx^{n-1} + \ldots$), even trigonometric or exponential functions. Furthermore, the fitted curve may only be an approximation, or it may be exact. Approximations are useful in correlating real data (the line of best fit, tidal predictions, population growth). Exact fits are useful in predicting the next number(s) in a given pattern, and we will be concentrating on exact fits here.

5. Find a function for which $f(0) = 10$, $f(1) = 7$, $f(2) = 10$, and $f(3) = 19$. Use any method. Then, use your new function to predict $f(5)$. Is there more than one function with these values as outputs?

There are many methods of solving these types of problems, which are frequent in Algebra II and Precalculus curriculums (find the line/parabola/exponential curve which passes through…). The one most frequently taught is to set up a system of equations; with a parabola, since its equation is of the form $f(x) = Ax^2 + Bx + C$, there would need to be three equations. We'll look at three other methods (if time permits, and hopefully it will!). The first is the method of common differences.

The method of common differences is useful when the value of the function is known for consecutive inputs (as it was in Problem 5). It works a little like one method to demonstrate slope – as you add 1 to the value of x, the value of y grows by a certain number. For a linear function, the common differences will be constant. This is a typical method of teaching the concept of slope, since the common difference represents the change in y, which is then divided by the change in x.

6. List the common differences for the function table given below. In the Δ column, use the value of $f(x+1) - f(x)$. (See the example: $9 = 19 - 10$.) Note there will be no Δ value next to $f(3)$, since we are not given $f(4)$.

x	$f(x)$	Δ
0	10	
1	7	
2	10	9
3	19	

7. What is suggested by the table of differences, as far as the type of the function $f(x)$ might be? Can it be a line? Why or why not? Can it be a quadratic?

The table of differences can be continued, indefinitely, so long as you have enough inputs to work with. For now, let's assume the table continues without any "surprises."

8. Continue the table in Problem 6 up to $f(5)$. Try to do this without relying on the formula you found for $f(x)$, but any method is fine.

We can also continue the table horizontally by writing the second differences, the third differences, and the fourth differences.

9. Continue the table below (see the examples if you need guidance). How far to the right could this table be continued?

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
0	10	-3	6	0	
1	7	3	6		
2	10	9			
3	19				
4	34				
5	55				

So long as the original function is a polynomial, and you are given consecutive inputs, the method of common differences can be used to determine the degree of the polynomial.

10. Give a rule that would let you determine, using common differences, the degree of a polynomial.

Sure, the degree is nice, but the coefficients would be nicer.

11. For the function $f(x) = Ax^2 + Bx + C$, build a table of common differences for $x = 0$ through 5. To get you started, $f(0) = C$, $f(1) = A + B + C$, $f(2) = 4A + 2B + C$, etc. Continue the table until you reach a point at which the common differences are constant. A calculator with a CAS (Computer Algebra System), like the TI-92, may be helpful.

x	$f(x)$	Δ	Δ^2	Δ^3
0	C	$A+B$		
1	$A+B+C$			
2	$4A+2B+C$			
3	$9A+3B+C$			
4				
5				

12. Use what you have learned so far to find the quadratic function for which $f(0) = 10$, $f(1) = 14$, $f(2) = 22$, $f(3) = 34$, and $f(4) = 50$. How many points do you need to be given to find the function, if you know it is quadratic?

13. Use the common differences to find the smallest degree possible for a polynomial that has $f(0) = 1$, $f(1) = 4$, $f(2) = 15$, $f(3) = 40$, $f(4) = 85$, $f(5) = 156$, $f(6) = 259$. Then, try to find the coefficients of this polynomial. How many functions are there with these values for $f(x)$?

14. Look across the first row in the table for Problem 9. Say you are given all the information for that row, and only that information. Could you construct the rest of the table? How? Specifically, how would you use the numbers in the first row to generate $f(5)$?

Problem 14 leads to an unusual and amazing way to do these problems; believe it or not, it has to do with Pascal's Triangle. If you are curious, you might try this method on Problems 13 and 24.

So the method of common differences works very well when you know consecutive values of the input, but what if you don't? In this case, a method known as *Lagrange Interpolation* can be used. It relies on an important fact:

If $f(a) = 0$, then $(x - a)$ divides $f(x)$. (In other words, $(x - a)$ is a $factor$ of $f(x)$.)

This result is used in Algebra II to find factored polynomials. It is also the reason why synthetic division works. For example, if we know a polynomial function has $f(4) = 0$ and $f(-3) = 0$, then two of its factors must be $(x - 4)$ and $(x + 3)$. Note that while we know these are factors, we do *not* know if there are others, unless we also know the degree of the polynomial.

15. Find the one quadratic function for which $f(0) = 36$, $f(2) = 0$, and $f(6) = 0$. Think of a factored form of the quadratic first, which will include $(x - 2)$ *and* $(x - 6)$ as factors. Then, expand the factored form, or just leave it in its factored form.

16. Find the one quadratic function for which $f(0) = 0$, $f(2) = 36$, and $f(6) = 0$.

17. Use the results from Problems 15 and 16 to find a function for which $f(0) = 36$, $f(2) = 36$, and $f(6) = 0$. If you're not using the results from 15 and 16... well, you should be!

18. Does Problem 17 remind you of anything? Hmmmmm?

Once you understand the method for Problem 17, it extends very quickly. Say we have a table:

x	$f(x)$
0	4
2	26
5	194
-1	2

We can build a solution which will be a third-power polynomial (why?), although it is possible that the resulting polynomial will have lower degree (if the leading coefficients come out to be zero). What we'll do is construct a polynomial that has this table:

x	$f(x)$
0	1
2	0
5	0
-1	0

Such a polynomial as form $A(x-2)(x-5)(x+1)$ because of the theorem above, and furthermore, we can find A by noting that $f(0) = 1$. This gives $A = \frac{1}{10}$, so the overall function is $\frac{1}{10}(x-2)(x-5)(x+1)$, and we will call this $A(x)$.

19. Continue this process to find $B(x)$, $C(x)$, and $D(x)$.

Note that each of these are third-power functions (since they each have three distinct factors), and also note that we have not yet used the original values of $f(x)$ at all, just the inputs we were given.

Now comes the Chinese part.

20. Use $A(x)$ through $D(x)$ to write a polynomial that satisfies the condition of the table above, then use a CAS to evaluate.
21. Find a function for which $f(0) = -10$, $f(2) = 6$, $f(5) = 15$, and $f(-1) = -9$. What steps in Problem 19 did you need to repeat?
22. Use Lagrange Interpolation to find a function for which $f(0) = -5$, $f(1) = -6$, $f(2) = 7$, $f(3) = 166$, $f(4) = 843$, and $f(5) = 2770$. You will definitely want to use a CAS for this. How does the answer compare to the answer given by common differences?
23. Use the methods we've discussed to find a function for which $f(x)$ is equivalent to $2x + 5$ for all integers from $x = 0$ through 3, but fails to match at any other value of x. Such a polynomial function would have to have at least what degree?
24. What about $f(x) = 2^x$? Will any of our methods be useful? Is this at all related to power series?
25. Try Problem 11 for cubic functions, for exponential functions. The laws of exponents will come in handy if you try exponential functions.
26. We said earlier that the Chinese Remainder Theorem only works for things that are relatively prime. Are $(x-2)$ and $(x-6)$ relatively prime? Are they prime themselves? It's important here to remember that x represents all numbers, so a particular value of x would not be useful to claim that they are not relatively prime.