

Session 13, July 25

Applications of Gaussian Integers

1. Use the division algorithm in $\mathbf{Z}[i]$ to find a valid quotient and remainder when $(5 + 7i)$ is divided by $(3 + i)$.
2. Use the Euclidean algorithm in $\mathbf{Z}[i]$ to find a GCD of $(5 + 7i)$ and $(3 + i)$. How is it possible for there to be more than one GCD?

A lattice structure can be used to picture the division algorithm in $\mathbf{Z}[i]$ (see handout).

3. Use a lattice on graph paper to find the quotient and smallest possible remainder when $(8 + 12i)$ is divided by $(3 + 2i)$.

Recall that the norm of a Gaussian integer $z = (a + bi)$ is $N(z) = a^2 + b^2$, and the norm is the square of the magnitude (or modulus) of that Gaussian integer.

4. Explain why the norm must be an integer. Can the norm ever be a perfect square?
5. Show that if two Gaussian integers are multiplied, their norms are multiplied. In other words, given two complex numbers x and y , show that $N(xy) = N(x)N(y)$.

One consequence of Problem 5 is that the square of any Gaussian integer must have a square number as its norm.

6. For each of these Gaussian integers, plot the Gaussian integer, then find and plot its square: $(1 + i)$, $(2 + i)$, $(3 + 2i)$, $(1 + 4i)$, $3, 2i$. Graphically, what relationships are there between a Gaussian integer and its square?
7. For each of the square Gaussian integers in Problem 6, construct a right triangle (if possible) with the square as the hypotenuse. What can be said about this triangle? What happens in those cases where a right triangle cannot be constructed?

A second consequence of Problem 5 is that the norm of a square Gaussian integer allows the equation $a^2 + b^2 = c^2$ to be solved with integers for a , b , and c . The values for a and b are the real and imaginary parts of the Gaussian square, and the value for c is the magnitude.

So, one way of finding Pythagorean triples is to start with any Gaussian integer, square it, and yabba dabba doo, there's a triple.

1. Generate a Pythagorean triple from the Gaussian integer $3 + i$. Generate a triple from $7 + 4i$. Generate a triple from $22 + 19i$. Generate a triple from $8 + 11i$.
2. Generate a Pythagorean triple from the Gaussian integer $(m + ni)$. Note that this will give you a general formula for finding an unlimited plethora of Pythagorean triples.

One issue which you may have noticed is that the Pythagorean triples are not always "primitive"; that is, the numbers a , b , and c are not relatively prime. Additionally, multiplication of Gaussian integers makes it possible for a , b , or both to become negative; not a good thing for the sides of a triangle.

10. What must be true of m and n if $(m + ni)$ generates a primitive Pythagorean triple? Alternatively, what must be true of m and n if $(m + ni)$ does *not* generate a primitive triple?
11. If m and n are both positive, what must be true of m and n for all the numbers in the generated triple to be positive? Your work in either Problem 6 or 9 can be used here.
12. Use multiplication of Gaussian integers to find all Pythagorean triples (not just primitives!) with hypotenuse $c = 25$; with $c = 35$; with $c = 65$; with $c = 125$. Use the property from Problem 5 here!
13. Which prime numbers can be the hypotenuse of a Pythagorean triple? Are all these triples primitive?
14. Which prime numbers can be the norm of a Gaussian integer?
15. What are the prime numbers in $\mathbf{Z}[i]$? Plot them in the plane, up to norm 50.

One of the reasons the process for finding Pythagorean triples works is because of factoring in $\mathbf{Z}[i]$:

$$\begin{aligned}c^2 &= a^2 + b^2 \\c^2 &= (a + bi)(a - bi)\end{aligned}$$

If such a factoring is possible, then the norm of $(a + bi)$ must equal c^2 ; and c is the length of the third side of the triangle.

A similar factoring allows for the generation of *Eisenstein triples*, which are numbers which form the sides of a triangle with a 60-degree angle. The Law of Cosines gives us the relationship between the three sides:

$$c^2 = a^2 + b^2 - 2ab \cos 60 \tag{1}$$

$$c^2 = a^2 + b^2 - ab = N(c) \tag{2}$$

(Note: the right side of (2) is the quadratic factor of $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.)

Factoring gives $c^2 = (a + b\alpha)(a + b\bar{\alpha})$, with the new number α having the property that $\alpha^2 + \alpha + 1 = 0$. (This is similar to i , where $i^2 + 1 = 0$.) Note that $\bar{\alpha}$ is the conjugate of α . It is more typically written that $\alpha^2 = -\alpha - 1$ since this form is more useful in algebraic simplification.

It takes a little getting used to this new number, although α is actually a member of \mathbf{C} . (Specifically, it is the number $\text{cis } 60$). α is used in the best-known proof that there are no solutions to the equation $x^3 + y^3 = z^3$. Numbers in the form $z = a + b\alpha$, where a and b are integers, are written as $\mathbf{Z}[\alpha]$.

16. Verify, using the property of α , that the factoring $c^2 = (a + b\alpha)(a + b\bar{\alpha})$ is valid.

Now, generating Eisenstein triples is done *the exact same way* we generated Pythagorean triples. Choose any element in $\mathbf{Z}[\alpha]$ and square it. Don't forget that the property of α is not the same as the property of i .

17. For each of these, find the square and the corresponding Eisenstein triple:
 $(1 + \alpha), (2 + \alpha), (3 + 2\alpha), (1 + 4\alpha), 3, 2\alpha$.
18. Generate an Eisenstein triple from the number $(m + n\alpha)$. This will give a general formula to generate Eisenstein triples.

As with Pythagorean triples, generating primitive triples becomes an issue, as does generating triples which have positive legs.

19. Describe conditions on m and n which will lead to primitive Eisenstein triples with positive side lengths.
20. Which prime numbers can be the “hypotenuse” (side opposite the 60-degree angle) of an Eisenstein triple? Of a *primitive* Eisenstein triple?
21. Beginning with the Law of Cosines, find a formula that will generate triangles with one angle of 120 degrees.