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Generation X and Y

1. Experiment with the following function using your CAS (for instance try $n = 0, 1, 2, 3$):

$$\frac{1}{2}(t + \sqrt{t^2 - 1})^n + \frac{1}{2}(t - \sqrt{t^2 - 1})^n$$

The generating function for the recursion
$$\begin{cases} P_0 = 2 \\ P_1 = 1 \\ P_n = P_{n-1} + 6P_{n-2} \end{cases}$$

is $\frac{2-x}{1-x-6x^2}$. The power series produced by the generating function is $2 + x + 13x^2 + 19x^3 + 97x^4 + 211x^5 + \dots$. When it is decomposed into partial fractions, it gives $\frac{1}{1-3x} + \frac{1}{1+2x}$. From this, we can see that the n^{th} coefficient of these two generating functions is $3^n + (-2)^n$.

$3^n + (-2)^n$ is the “closed” formula that produces the n^{th} coefficient of the power series produced by the generating function

$\frac{2-x}{1-x-6x^2}$ and is the n^{th} term in the recursion
$$\begin{cases} P_0 = 2 \\ P_1 = 1 \\ P_n = P_{n-1} + 6P_{n-2} \end{cases}$$
.

2. Check to see that $3^n + (-2)^n$ actually produces the power series above.

PARTIAL FRACTIONS (one last time)

In general, $\frac{a}{xy}$ can be written as $\frac{A}{x} + \frac{B}{y}$. Solve a system of equations for A and B to get the answer(s).

3. The generating function of the recursion is listed below the recursion. The power series produced by the generating function is below that. Break the generating function into partial fractions and

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find the “closed” formula the n^{th} coefficient of the power series produced by the generating function.

$$\begin{aligned} B_0 &= 2 \\ B_1 &= 10 \\ B_n &= 10B_{n-1} + 21B_{n-2} \end{aligned}$$

$$\frac{2 + 10x}{1 - 10x + 21x^2}$$

$$2 + 10x + 58x^2 + 370x^3 + 2482x^4 + \dots$$

4. Repeat the directions from problem 3.

$$\begin{aligned} C_0 &= 0 \\ C_1 &= 4 \\ C_n &= 10C_{n-1} + 21C_{n-2} \end{aligned}$$

$$\frac{4x}{1 - 10x + 21x^2}$$

$$0 + 4x + 40x^2 + 316x^3 + 2320x^4 + \dots$$

5. Repeat the directions from problem 3.

$$\begin{aligned} F_0 &= 4 \\ F_1 &= 26 \\ F_n &= 12F_{n-1} + 35F_{n-2} \end{aligned}$$

$$\frac{4 + 22x}{1 - 12x + 35x^2}$$

$$4 + 26x + 172x^2 + 1154x^3 + \dots$$

6. Find the roots of the denominator in problem 3 (you can just use the quadratic formula). Compare these roots to the “closed” formula you obtained for the n^{th} coefficient of the power series produced by the generating function? If you know the denominator of the generating function, do you know something that is in the “closed formula”?

Answer these same questions for problem 4 and for problem 5.

For any quadratic (ax^2+bx+c), we will refer to one of its roots as r_1 and to its other root as r_2 .

Most generating functions with a quadratic in the denominator will have a “closed” formula for the n^{th} coefficient of the power series produced by the said generating function. It will be of the form $c(j)^n \pm d(k)^n$, where c and d are constants.

Again, your CAS will do this for you. Go to “zeros” under the “F2-Algebra” menu, number four. **zeros(1-x-6x²,x)** will give you the roots for the denominator $1-x-6x^2$.

For any quadratic we will refer to one of its roots as r_1 and to its other root as r_2 .

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7. For problem 3, show that $j = \frac{1}{r_1}$ and $k = \frac{1}{r_2}$. Do the same for problem 5. You've now shown it to be true twice, now you can consider it to be true for all cases.

Thus, the "closed" formula for the n^{th} coefficient of the power series produced by a generating function with a quadratic in

its denominator is $c \frac{1}{r_1} \pm d \frac{1}{r_2}$

Alice and Bob's method for generating a "closed" formula from the roots of the denominator (a.k.a. **The Root Theorem**):

1. Get the generating function for your recursion.
2. Find the roots of the denominator of the generating function.
3. Set $c \frac{1}{r_1} + d \frac{1}{r_2}$ equal to F_0 . (You know F_0 from the original recursion.)
4. Set $c \frac{1}{r_1} + d \frac{1}{r_2}$ equal to F_1 . (You know F_1 from the original recursion.)
5. Solve two equations for two unknowns (you now have c and d).

8. Apply **The Root Theorem** to problem 4 to show yourself that it works. Did it work? If so, it always works. (Do problem 5 if you must do something twice to consider it proved).

Maybe **The Root Theorem** should have been called the: "Now I don't need to use partial fractions ever again theorem."***

***This view is not necessarily reflective of the view of all of PCMI and should be treated as such.

Herbert explains to Alice and Bob that their theorem works "thanks to the magic of the geometric series."
Ask Jo Ann if she can do a magical song and dance to describe the geometric series.

Remember that you can always check your "closed formula with the values given by the recursion or by applying the CAS's "taylor" function to the generating function.

Fibonacci Numbers (0,1,1,2,3,5,8,...)

9. Use **The Root Theorem** to find the function that generates the n^{th} Fibonacci number.

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

$$\frac{x}{1 - x - x^2} = 0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + \dots$$

Lucas Numbers (2,1,3,4,7,11,18,...)

10. Use **The Root Theorem** to find the function that generates the n^{th} Lucas number.

You have to figure out the generating function (or at least the denominator).

$$\begin{aligned} \square L_0 &= 2 \\ \square L_1 &= 1 \\ \square L_n &= L_{n-1} + L_{n-2} \end{aligned}$$

$$2 + 1x + 3x^2 + 4x^3 + 7x^4 + \dots$$

Lucas numbers are formed by the same recursion as Fibonacci numbers, but start with 2 and 1.

11. Suppose the post office has 1 cent stamps and two kinds of 2 cent stamps. How many ways are there to get n cents? This time, which one of the 2 cent stamps you choose matters and so does the order, so we want permutations. Do this up for $n=1,2,3,4$. Can you find a recursive formula that fits your data? Think first, then look at the hint.

The post office has many kinds of 37 cent stamps. There is the “love” stamp and the “American flag” stamp. Maybe there should be a “fractal” stamp???

12. Use **The Root Theorem** to find a “closed” formula for the n^{th} term of your recursion.

Chebyshev Polynomials

$$C_0 = 1$$

$$C_1 = t$$

$$C_2 = 2t^2 - 1$$

$$C_3 = 4t^3 - 3t$$

$$C_4 = 8t^4 - 8t^2 + 1$$

Hint: 5 cents can be made 21 ways. 6 cents can be made in 43 ways. 7 cents can be made in 85 ways.

12. Use **The Root Theorem** to find the function that generates the n^{th} Chebyshev polynomial. Find the denominator of the generating function (or the whole generating function if you are glutton for punishment).

$$\begin{aligned} \square C_0 &= 1 \\ \square C_1 &= t \\ \square C_n &= 2tC_{n-1} - C_{n-2} \end{aligned}$$

$$1 + tx + (2t^2 - 1)x^2 + (4t^3 - 3t)x^3 + (8t^4 - 8t^2 + 1)x^4 + \dots$$

13. Experiment with the following matrices. Notice anything interesting?

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} * \begin{bmatrix} 8 \\ 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}^n$

14. Using matrices, can you say anything or prove anything about the n^{th} Fibonacci number?

15. Make a matrix or matrices that produce the Lucas numbers.

16. Experiment with the following matrices (determinants are the way to go here):

(a) $\begin{bmatrix} x & 1 \\ 2x & 1 \end{bmatrix}$ (b) $\begin{bmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{bmatrix}$ (c) $\begin{bmatrix} x & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{bmatrix}$

17. Find a set of matrices that will produce the solutions to yesterday's stamp problem.

17. Find the recursion and the "closed" formula for the n^{th} coefficient of the power series produced by: $\frac{2 + 6x}{1 + 6x + 9x^2}$