2007.1 Head Games

Hey, welcome to the class. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How did others at your table think about this question?
- **Teach only if you have to.** You may feel the temptation to teach others at your table. Fight it! We don't mean you should ignore your tablemates but give everyone the chance to discover. If you think it's a good time to teach your tablemates about integration by parts, think again: the problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology used in the course. The problems should lead to the appropriate use of technology rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Game of the Day, Important Stuff, Neat Stuff, and Tough Stuff. We'll have you do Game of the Day first thing (most of the time), but after that you should check out Important Stuff first. We try to make sure that the problems in Important Stuff can be picked up regardless of how much or little work you've done on prior sets. All the mathematics that is central to the course can be found and developed in the Important Stuff. Hey, it's important stuff. Everything else is just neat or tough.

One question from a previous year turned out to be the unsolved Twin Primes Conjecture. Nobody got that one.

On Day 3, go back and read these again.

Will you remember? Maybe!

And now, on with the show . . .

Game of the Day: "Fake It, Make It"

1. With a partner or two, *fake* the results of flipping a coin 240 times. Write heads as 1, tails as 0. Seriously, fake it: no technology, no dice, nothing.

2. Now, with the same partnerortwo, *make* the results of flipping a coin 240 times. Write heads as 1, tails as 0. Seriously, make it: no technology, no dice, just flip a coin 240 times and write down what it says. Identify the two lists in some way that you would recognize, but a neighbor would not.

3. Create a test you could use to decide whether a list someone gave you was real or fake. Now exchange lists with a neighbor and run your test. Did it work?

Important Stuff.

4. (a) Pick two integers between 1 and 5 (inclusive). Find the probability that the two numbers picked do not share a common factor greater than 1.

(b) Repeat for picking between 1 and 6, 1 and 7, 1 and 8, 1 and 9.

Be clear about any assumptions made in how you chose to do the picking—there is more than one way to do it, and they will yield different results that we'll talk about.

5. (a) If you flip a fair coin 240 times, how many heads would you expect?

(b) Take a guess: what is the probability of getting *exactly* this many heads?

Don't take more than 10 seconds to guess!

6. (a) If you roll a fair die 240 times, how many ones would you expect?

(b) Another guess: is it more likely to get exactly this many ones, or to get exactly the number of heads from the last problem?

Dude, just take a guess and move on.

7. Jocelyn has a piece of paper. She tears it into three equal pieces and hands one piece to Connie, another to Mario, and keeps the third piece for herself.

She continues to do this; tearing the paper she has left into three equal pieces, handing one piece to Connie, one to Mario, and keeping the third.

(a) After two tears, how much paper does Jocelyn have left? How much do Connie and Mario each have?

(b) After three tears?

(c) After four tears?
(d) After 10 tears?
(e) Forever?
(f) Write two different expressions for the amount of paper Connie has after this is over.

8. Mary has a piece of paper. She tears it into four equal pieces and hands one piece to Manuel, one piece to Kim, one piece to Sandra, and keeps the fourth piece for herself.

She continues to... aw, do we have to write the rest of this? Fine.

(a) After two tears, how much paper does Mary have left? How much do Manuel, Kim, and Sandra each have?
(b) After three tears?
(c) After four tears?
(d) After 10 tears?
(e) Forever?
(f) Write two different expressions for the amount of paper Sandra has after this is over.

Neat Stuff.

9. What's the probability that an integer picked from 1 to n is a perfect square if

(a) $n = 10$?
(b) $n = 100$?
(c) $n = 1000$?
(d) $n = 10000$?
(e) What is happening "in the long run" (as n grows larger without bound)?

10. A gambler offers you these two games:

Game 1: You roll a die four times. If you hit a 1 any of the four times, you win.

Game 2: You roll a pair of dice 24 times. If you hit double 1s any of the 24 times, you win.

Aside from the length-of-game issue, which of these games would you rather play to win? Or are they the same?

11. The new toy craze is Mega Men, where kids buy a Mega Man in a box without looking to see which one it is, then open it up when they get home. There are ten toys in all, each equally likely when you buy a box. If you collect all ten, you can make Ultra Mega Mega Man!

On average, how many boxes will you have to buy for your kid before he can finally collect them all?

Sadly, eBay is not an option, since the only cool Mega Men are the ones still in their original packaging.

12. Okay, so there's this game. You get 1 point every time you flip heads. But, anytime you flip tails you're in "danger". If you flip tails a second time *consecutively*, you "bust" and lose all your points (but continue playing).

So, don't flip tails twice in a row. Otherwise it's all good.

(a) The game lasts 10 flips. What is the probability that you survive all 10 flips without busting even once?

(b) What is the average score you could expect after 10 flips?

(c) What happens in a longer game? Will the average score increase? Is there a limit?

Tough Stuff.

13. Remember Yahtzee? Sure, you remember Yahtzee. You get three rolls and you're looking to get all 5 dice to be the same number. You can "save" dice from one roll to the next. There are other goals, but people really only care about getting the 5-dice Yahtzee.

(a) Find the probability that if you try for it, you will get a Yahtzee of all 6s in any given turn.

(b) Tougher: find the probability of getting any Yahtzee by trying for it—that is, you always play toward the nearest available Yahtzee.

14. Build a data set with at least 5 elements such that if m is the mean and n is the median, then $|m - n|$ is larger than the standard deviation of the set.

Since this is Tough Stuff, we don't have to tell you how to calculate standard deviation.

2007.2 Games People Play

Game of the Day: "Bingo Hi-Lo"

Bill is a contestant on a game show. On this show, an integer between 1 and 75 is picked (at random). Then you are asked whether the next number (picked from the *remaining* numbers) will be higher or lower. If you are correct, you'll be paid \$100 times the number that comes out. Your plan is to make the decision that makes as much money as possible.

For example, if the first number is 25 and you say "Higher", and the next number is 35, you'd win \$3500. Easy money!

1. (a) Say the first number is 25. What should you do?
(b) Okay, first number 38. What should you do?
(c) Okay, first number 42. What should you do?
(d) So, what's your overall strategy? State it clearly.

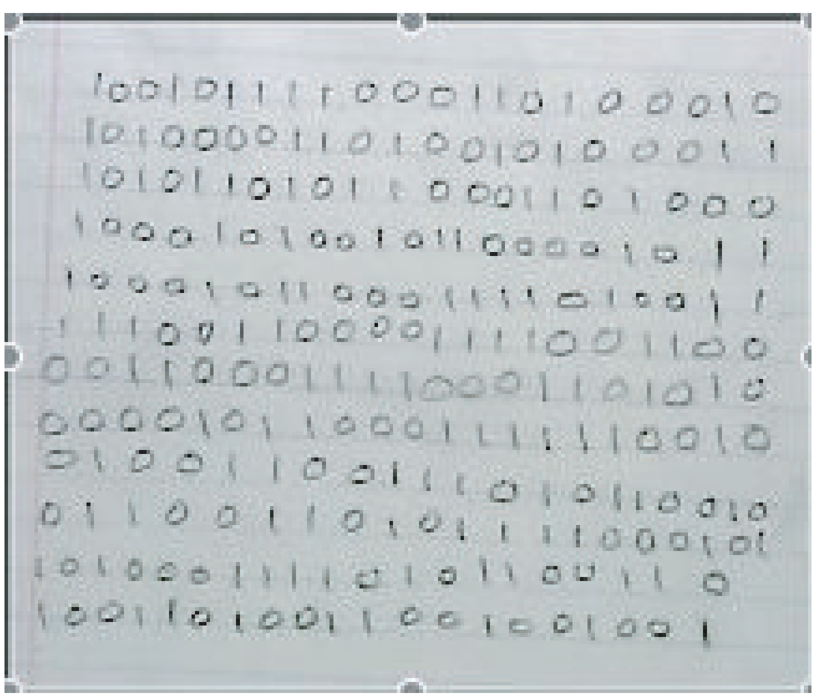
Important Stuff.

2. Describe, in complete detail, a test you could perform on a set of 240 coin flips that would help you decide whether it is real or fake.

3. *Use your test* on each of these four data sets to decide whether each is real or fake.

They could all be fake; or they could all be real; or . . . If you want more of these, we have them and will happily give you another page of them.

(a)



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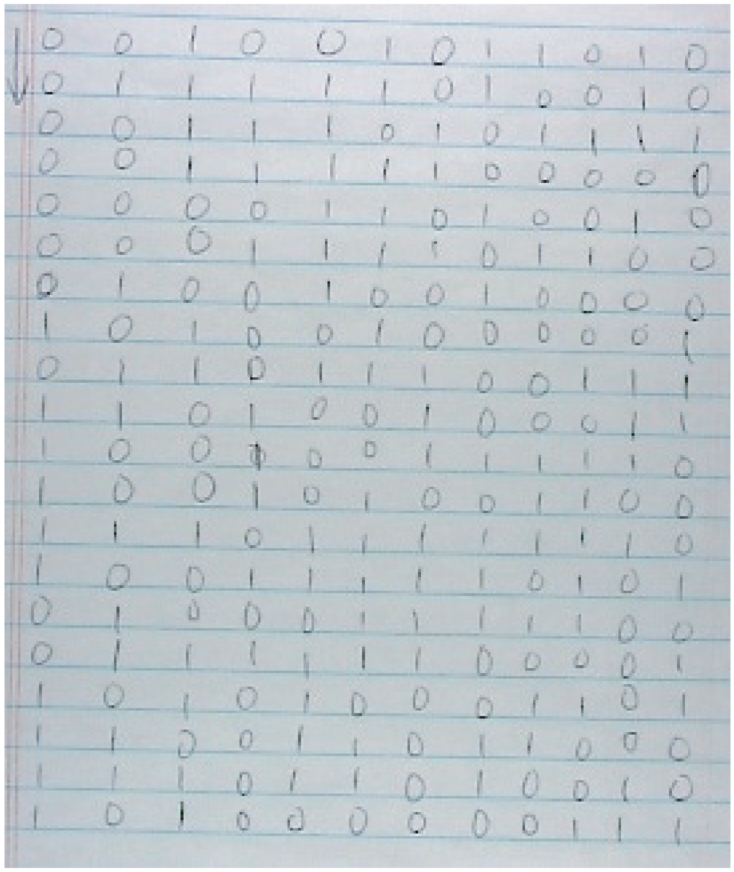
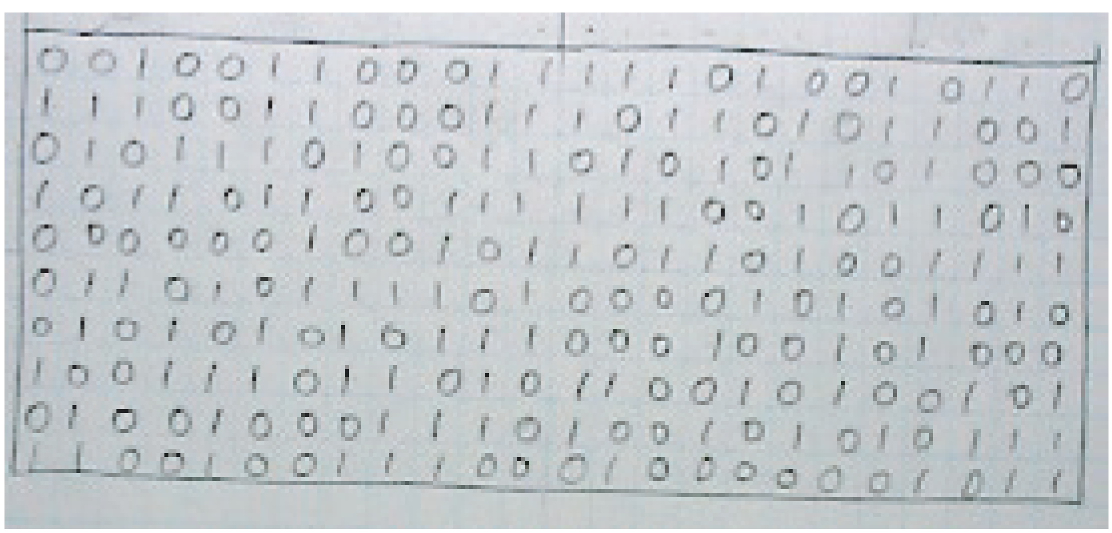


This one is read vertically; see the arrow. Go down each column. The others are read horizontally.

(c)



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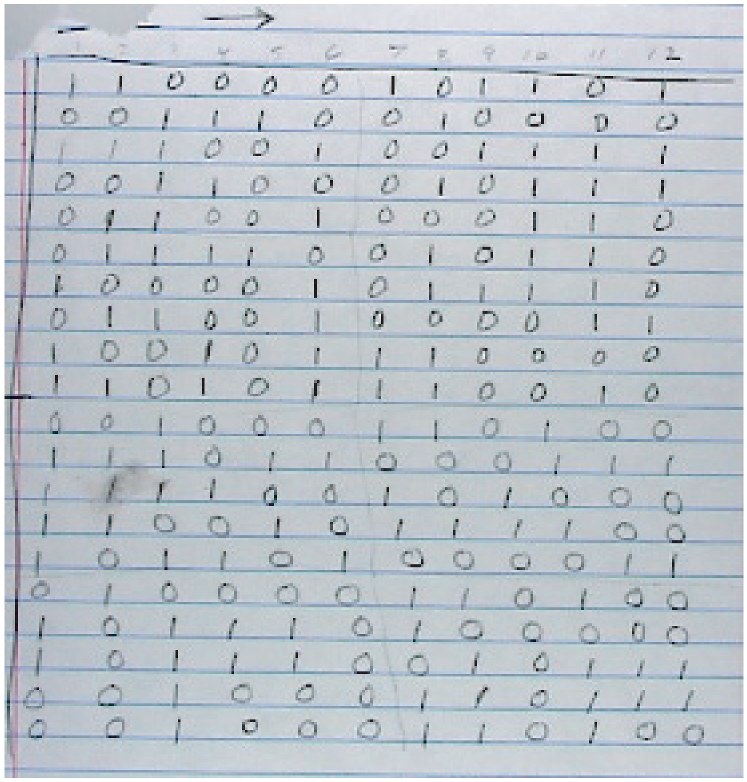


4. (a) Pick two integers between 1 and 6 (inclusive). Find the probability that the two numbers picked do not share a common factor greater than 1. There is more than one possible correct answer.

(b) Repeat for picking between 1 and 7, 1 and 8, 1 and 9, 1 and 10.

5. (a) If you flip a fair coin 240 times, how many heads would you expect?

(b) Take a guess: what is the probability of getting *exactly* this many heads?

Don't take more than 10 seconds to guess! Make sure your table has a conversation about this at some point if they haven't already.

6. (a) If you roll a fair die 240 times, how many ones would you expect?

(b) If you roll a fair die 240 times, what's the expected *sum* of the 240 die rolls?

7. Jocelyn has a piece of paper. She tears it into three equal pieces and hands one piece to Connie, another to Mario, and keeps the third piece for herself.

She continues to do this; tearing the paper she has left into three equal pieces, handing one piece to Connie, one to Mario, and keeping the third.

(a) After two tears, how much paper does Jocelyn have left? How much do Connie and Mario each have?
(b) After three tears?
(c) After four tears?
(d) After 10 tears?
(e) Forever?
(f) Write two different expressions for the amount of paper Connie has after this is over.

8. Mary has a piece of paper. She tears it into four equal pieces and hands one piece to Manuel, one piece to Kim, one piece to Sandra, and keeps the fourth piece for herself.

She continues to... aw, do we have to write the rest of this? Fine.

(a) After two tears, how much paper does Mary have left? How much do Manuel, Kim, and Sandra each have?
(b) After three tears?
(c) After four tears?
(d) After 10 tears?
(e) Forever?
(f) Write two different expressions for the amount of paper Sandra has after this is over.

Neat Stuff.

9. What's the probability that an integer picked from 1 to n is a perfect square if

(a) $n = 10$?
(b) $n = 100$?
(c) $n = 1000$?
(d) $n = 10000$?
(e) What is happening "in the long run" (as n grows larger without bound)?

10. A gambler offers you these two games:

Game 1: You roll a die four times. If you hit a 1 any of the four times, you win.

Game 2: You roll a pair of dice 24 times. If you hit double 1s any of the 24 times, you win.

Aside from the length-of-game issue, which of these games would you rather play to win? Or are they the same?

11. The new toy craze is Mega Men, where kids buy a Mega Man in a box without looking to see which one it is, then open it up when they get home. There are ten toys in all, each equally likely when you buy a box. If you collect all ten, you can make Ultra Mega Mega Man!

On average, how many boxes will you have to buy for your kid before he can finally collect them all?

Sadly, eBay is not an option, since the only cool Mega Men are the ones still in their original packaging.

12. You want to roll 15 as the sum of dice. How many dice do you pick to max your chances?

13. Okay, so there's this game. You get 1 point every time you flip heads. But, anytime you flip tails you're in "danger". If you flip tails a second time *consecutively*, you "bust" and lose all your points (but continue playing).

So, don't flip tails twice in a row. Otherwise it's all good.

(a) The game lasts 10 flips. What is the probability that you survive all 10 flips without busting even once?

(b) What is the average score you could expect after 10 flips?

(c) What happens in a longer game? Will the average score increase? Is there a limit?

Tough Stuff.

14. Remember Yahtzee? Sure, you remember Yahtzee. You get three rolls and you're looking to get all 5 dice to be the same number. You can "save" dice from one roll to the next. There are other goals, but people really only care about getting the 5-dice Yahtzee.

(a) Find the probability that if you try for it, you will get a Yahtzee of all 6s in any given turn.

(b) Tougher: find the probability of getting any Yahtzee by trying for it—that is, you always play toward the nearest available Yahtzee.

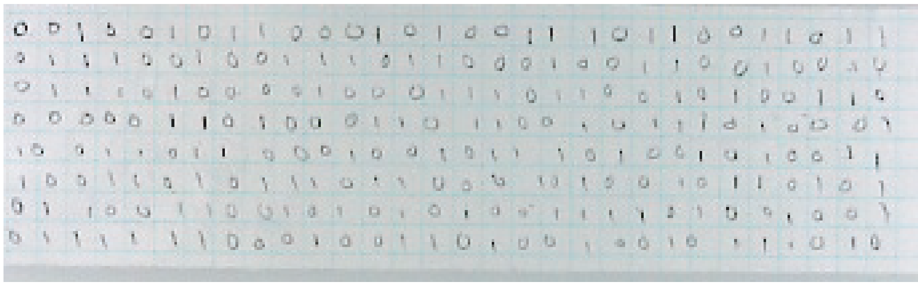
15. Build a data set with at least 5 elements such that if m is the mean and n is the median, then $|m - n|$ is larger than the standard deviation of the set.

Since this is Tough Stuff, we don't have to tell you how to calculate standard deviation.

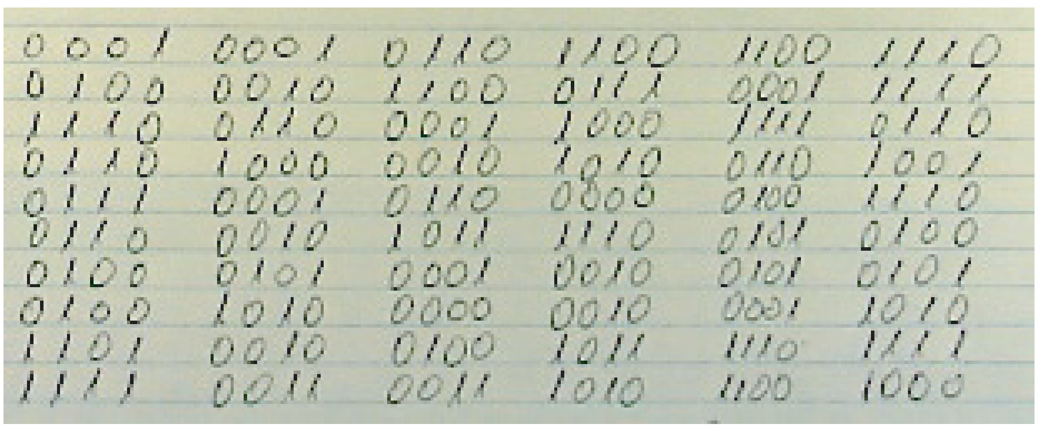
Three More Data Sets

If you want them.

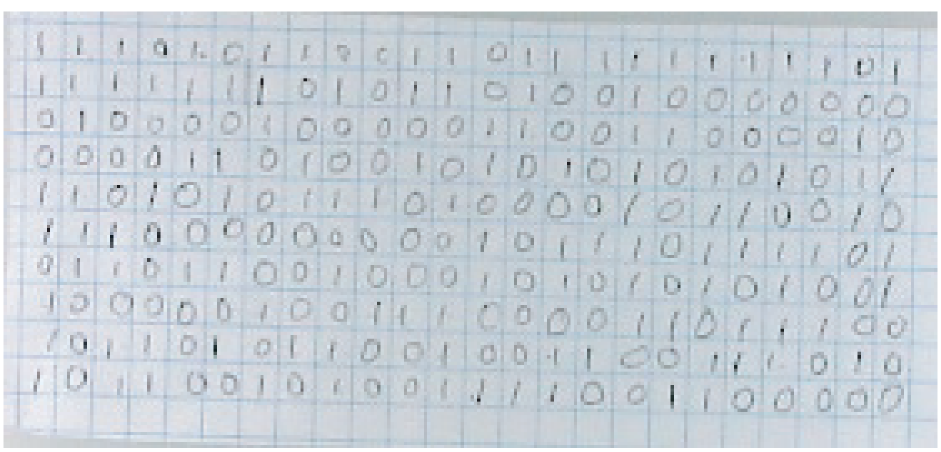
(e)



(f)



(g)



2007.3 Game On!

Game of the Day: "¿Vas o No Vas?"

A popular game show involves opening suitcases with different dollar amounts in them. Every so often, the player is offered a deal to walk away instead of continuing the game. If the player refuses all deals, they get to keep whatever is in the final suitcase.

1. There are four suitcases left: \$5, \$25, \$10,000, and \$50,000.

(a) The show offers you \$9,000 to walk away. Should you take the deal? Why or why not?

(b) What if the show offered you \$12,000? Would you take that deal? Why or why not?

2. There are five suitcases left: \$1, \$5, \$5,000, \$30,000, and \$125,000. You are a representative of the show. What deal amount would you offer, and why?

3. The actual show begins with these 26 suitcase values:

If you're expecting a \$1 million suitcase, you've got the wrong show.

\$.01	\$.25	\$1	\$5	\$10	\$25
\$50	\$75	\$100	\$200	\$300	\$400
\$500	\$1,000	\$1,250	\$2,500	\$5,000	\$7,500
\$10,000	\$25,000	\$30,000	\$40,000	\$50,000	\$100,000
\$125,000	\$250,000				

What would a *fair deal* be worth at the start of the game? Describe how the calculation was made.

Important Stuff.

4. This is G_5, the *Godmother sequence* of order 5:

This is opposed to the *Godfather sequence*, which is {I, II, III}.

$$\left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$

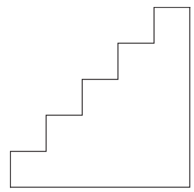
The sequence G_5 is all fractions from 0 to 1, inclusive, with denominators less than or equal to 5, and is written in increasing order with all fractions written in lowest terms.

Write out G_6, G_7, and G_8.

5. How many elements are in each of G_1 through G_8?

This follows up on some of the geometric discussion we had Tuesday.

6. Here is a geometric shape:



(a) If the length and height of each stair is 1 foot, find the area of the shape.

(b) Show how two staircases of this shape could be combined into a rectangle, and find the area of the rectangle.

(c) Suppose the staircase was 9 steps instead of 5. Could two such staircases still form a rectangle? Use this to find the area of *one* such staircase.

(d) Suppose the staircase was 75 steps. Could two such staircases still form a rectangle? Use this to find the area of *one* such staircase.

(e) Find a rule for the sum of the first n integers

$$1 + 2 + 3 + \cdots + n$$

7. Pick one of the coin sets from Tuesday that you *haven't* worked on yet, and determine whether it is real or fake. We are likely to have a discussion about the different tests people have been designing.

There were seven on Tuesday's set, three on an appendix page. We have more, just ask.

8. After the parade, Alan was stuck with a giant piece of paper. He tears it into five equal pieces and hands one piece to Caroline, one piece to Jim, one piece to Philip, one piece to RoseMary, and keeps the fifth piece for himself.

He continues to do this with his remaining piece, dividing it into five equal pieces.

(a) After two tears, how much paper does Alan have left? How much do the others each have?

(b) After three tears?

(c) After 10 tears?

(d) Forever?

(e) Write two different expressions for the amount of paper Caroline has after this is over.

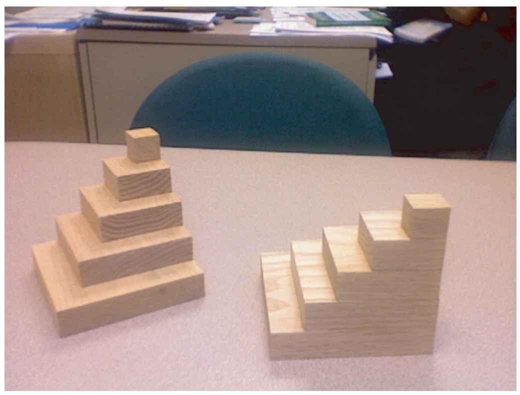
Neat Stuff.

9. On the hit show Italian Mathematical Bingo Night, the bingo machine is filled with balls for the Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233$$

How would the player's strategy in Tuesday's "Game of the Day" be altered by this change?

10. The formula for the sum of the first n squares can be modeled with blocks of wood! Here's a picture of two blocks.



(a) Explain why the volume of each block of wood pictured above is

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

(b) Show how six blocks in this shape can be fit together to form a solid box (or, better yet, build it). Then, find the dimensions of this box.

(c) Describe how this process could be generalized to show that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

11. What's the probability that an integer picked from 1 to n is a perfect square if

(a) $n = 10$?
(b) $n = 100$?
(c) $n = 1000$?
(d) $n = 10000$?

You'll notice a few retreads from earlier sets. Skip 'em if you already did 'em, but they're still neat if you haven't seen them yet.

(e) What is happening "in the long run" (as n grows larger without bound)?

12. A gambler from France offers you these two games:

Game 1: You roll a die four times. If you hit a 1 any of the four times, you win.

Game 2: You roll a pair of dice 24 times. If you hit double 1s any of the 24 times, you win.

Aside from the length-of-game issue, which of these games would you rather play to win? Or are they the same?

13. The new toy craze is Mega Men, where kids buy a Mega Man in a box without looking to see which one it is, then open it up when they get home. There are ten toys in all, each equally likely when you buy a box. If you collect all ten, you can make Ultra Mega Mega Man!

On average, how many boxes will you have to buy for your kid before he can finally collect them all?

Sadly, eBay is not an option, since the only cool Mega Men are the ones still in their original packaging.

14. Okay, so there's this game. You get 1 point every time you flip heads. But, anytime you flip tails you're in "danger". If you flip tails a second time *consecutively*, you "bust" and lose all your points (but continue playing).

So, don't flip tails twice in a row. Otherwise it's all good.

(a) The game lasts 10 flips. What is the probability that you survive all 10 flips without busting even once?

(b) What is the average score you could expect after 10 flips?

(c) What happens in a longer game? Will the average score increase? Is there a limit?

Tough Stuff.

15. Do the Yahtzee problem! This shorter version has saved almost 100 pieces of paper, which will all be cut into equal pieces later.

16. Build a data set with at least 5 elements such that if m is the mean and n is the median, then $|m - n|$ is larger than the standard deviation of the set.

Since this is Tough Stuff, we don't have to tell you how to calculate standard deviation.

17. The "Game of the Day" for Tuesday is a shorter description for the actual game. The player has 17 turns to earn 500 points playing the game (1 point times the ball number picked). How would you find the probability that a player would win this game?

2007.4 It's Not How You Play The Game, It's How Often You Win Or Lose

Game of the Day: "Two Heads Up"

Time limit: 10 minutes.

Here's three games.

Game 1: Flip two coins. If you get exactly two heads, you win.

Game 2: Flip three coins. If you get exactly two heads, you win.

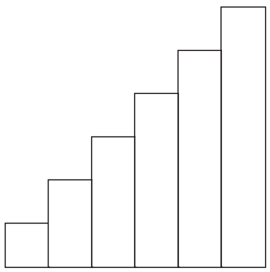
Game 3: Flip four coins. If you get exactly two heads, you win.

1. Which game gives you the highest probability of winning?

Next week is "The Price Is Right" week on Game of the Day. Be prepared.

Important Stuff.

2. (a) Calculate $1 + 2 + 3 + 4 + 5 + 6$.
(b) Here is a geometric shape Peg built from some rods.



If the width and height of each step is 1 cm, find the area of the shape.

3. Calculate the sum of the first 100 positive integers.

4. Take a coordinate grid with $0 \leq x \leq 10, 0 \leq y \leq 10$. Plot all points (x, y) in this range where x and y are both positive with $x \geq y$. Don't do it yet! If x and y share a common factor greater than 1, color them one way; if not, color them another way. Okay, now you can do it. It's safe. It's very safe.

Yes, use green and red! Yes, use pen and pencil! Just make it clear which is which.

5. Let x and y be two integers between 1 and 10, inclusive, with $x \geq y$. Use your graph from problem 4 to determine the probability that any such pair has no common factor greater than 1.

6. Write out F_{10}, the *Farey sequence* of order 10. Yesterday we called this the *Godmother sequence*. Today, John did a five-minute short on this sequence.

If this doesn't make sense, ask Flora, Fauna, or Meriweather. Or just watch Cinderella.

7. Take a coordinate grid with $0 \le x \le 10, 0 \le y \le 10$. Plot all points (x, y) in this range where the fraction $\frac{y}{x}$ is in the Farey sequence. Hmmm?

For example, you'd plot the point $(5, 3)$ since $\frac{3}{5}$ is in the Farey sequence. Don't plot $(3, 5)$, though.

8. Use your graphs to complete this table. Here, "relatively prime pairs" refers to the pairs plotted in problem 4 that have no common factors greater than 1. Some of the numbers have been included to aid you.

n	Number of elements in F_n	Relatively Prime Pairs
1	2	
2		
3		4
4		
5	11	
6		
7		18
8		
9		
10	33	

9. **Calculator skill time.** Here are some explicit directions on how to expand the expression $(h + t)^2$ on the calculators.

- Hit the HOME button then select option 5 to get a new document.
- When it asks you what kind of page you want, select Calculator.
- Type out the word `expand` then the expression you want to expand. There are two sets of parentheses, one for the `expand` function and one for the expression. The exponent button is on the left side two below CTRL. Your calculator line should look like

  `expand((h+t)`^{2}`)`
- Hit the enter key in the bottom right and enjoy the magic.

Use the green buttons: E-X-P...

(a) Expand $(h + t)^2, (h + t)^3$, and $(h + t)^4$.

(b) What does this have to do with the Game of the Day?

You could also hit CTRL then N for a new document. Lots of the typical command letters work the way they're supposed to.

Neat Stuff.

10. Use the expansion of $(h+t)^6$ to find the total number of ways you could flip 4 heads and 2 tails in a sequence of six coin tosses.

11. Pick a data set of 240 coin tosses, and lump them in 80 groups of three. For each group, count the number of heads: it will be zero, one, two, or three.

(a) How many of each category would you expect? There are 80 total groups.

(b) How does your data set compare? What might make you suspect a fake?

12. Find the *number of elements* in F_{30}, the Farey sequence of order 30. Try to find some shortcuts to help your work: your goal is only to find the number of elements, not list them all.

13. (a) Find the fraction with smallest integer denominator (and integer numerator) that is *between* $\frac{7}{17}$ and $\frac{5}{12}$.

(b) If the answer to the last question is $\frac{a}{b}$, find the fraction with smallest integer denominator (and integer numerator) that is *between* $\frac{7}{17}$ and $\frac{a}{b}$.

14. On the hit show Avery's Bingo Night, the bingo machine is filled with balls for the powers of 2:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048$$

How would the player's strategy in Tuesday's "Game of the Day" be altered by this change?

15. Today's "Game of the Day" asks you to find the best number of coins to flip to get two heads. What about three heads: what's the number of coins that gives you the best chance of flipping exactly three heads?

16. Draw a histogram with the number of coins flipped on the horizontal, and the probability of flipping exactly four heads on the vertical.

17. It's the weekend: surely you've missed some earlier Neat Stuff problems. If you're interested, check them out. They're neat—but remember that we'll make sure to cover anything really important in... well, yeah.

Tough Stuff.

18. The Price is Right game called "Spelling Bee" asks you to pick five cards from a board containing 30 cards:

11 cards say C
11 cards say A
6 cards say R
2 cards say CAR

Actually, the player needs to answer some easy questions about cutlery or MP3 players to earn the five cards, but we'll assume they get them all.

The goal is to spell CAR in any way possible: if you pull either of the 2 "CAR" cards, you win immediately. But you can also win by picking at least one of each of the three categories C, A, R.

Several video clips of this game are available on YouTube.

(a) What is the probability of winning this game?
(b) Remarkably, someone actually managed to spell CAR **three** times during the same game. How likely was *this*?

19. Henri and Tatyana play a very long game. They flip a coin: if it's heads, Henri gets a point. If it's tails, Tatyana gets a point. What makes it such a long game? Well, in order to win, you have to be 20 points ahead of your opponent.

How long, on average, will this game last?

20. A person is standing at the edge of a pool, and they've had one too many. Each step they take, they have a $\frac{1}{3}$ chance of stepping toward the pool, and a $\frac{2}{3}$ chance of stepping away from the pool.

What is the exact probability that they eventually fall into the pool? Note that this probability will be more than $\frac{1}{3}$ since the first step could take them into the deep end.

21. We've chosen not to repeat any problems on this set. So, if you want some other challenges, go back and look at the other Tough Stuff problems. We'd love to see a few people come back next week with a solution to the Yahtzee problem, or a data set where the difference between the mean and median is greater than the standard deviation.

2007.5 We've Got Game

Game of the Day: "Race Game"

Alice, Bev, Craig, and Dawn sit at a table for four in no particular order. Rey, the guest of honor, tells them their seats are assigned at the table and shows them the chart.

Rey is getting an award as the King of Math.

Twelve minute time limit!

1. (a) What is the probability that *all four* of them are in the right seat?
(b) What is the probability that *all four* of them are in the *wrong* seat?

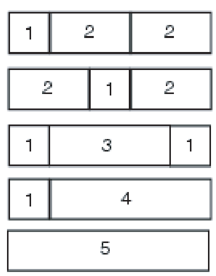
2. *On average*, how many of the four will be sitting in the right seat?

Important Stuff.

You can use rods of integer sizes to build "trains" that all share a common length.

Using Cuisenaire rods helps. Unless you are told otherwise, you have an unlimited supply of all the rod types, even beyond the 10 that is the normal limit of the Cuisenaire set.

A "train of length 5" is a row of rods whose combined length is 5. Here are some examples:



Notice that the 1-2-2 train and the 2-1-2 train contain the same rods but are listed separately. If you use identical rods in a different order, this is a distinct train.

3. (a) How many distinct trains of length 4 can be made?
(b) How many trains of length 4 can be made with one rod? two rods? three? four? five? A 1-1-2 train counts as three rods, by the way.

Five rods? Eh? Let's just say this shouldn't be that hard.

4. Repeat problem 3 for trains of length 5. The work on the previous problem might be helpful!

5. (a) Make a table for $n = 2$ to 8 for the number of trains of length n that use *exactly* two rods.
(b) Repeat for *exactly* three rods.

6. Physically make all five trains of length 6 that use exactly two rods, and (separately, without destroying the two-rod trains) all ten trains of length 6 that use exactly three rods.

7. **Calculator skill time.** Here are some explicit directions on how to evaluate $\binom{5}{2}$ on the calculators.

- Hit the HOME button then select option 5 to get a new document.
- When it asks you what kind of page you want, select Calculator.
- Type out the word `ncr` then, in parentheses: `(5, 2)`. Your calculator line should look like this:

  `ncr(5,2)`
- Hit the enter key in the bottom right and enjoy the magic. The letter C will be capitalized in the display, which will appear as `nCr(5,2)`.

(a) Calculate $\binom{5}{2}$, $\binom{6}{2}$, and $\binom{7}{2}$.
(b) The calculator can work with variables, too. What does the calculator give for $\binom{x+1}{2}$?
(c) What does the calculator give for $\binom{x}{k}$?

Not sure what $\binom{5}{2}$ means? No worries, we shall discuss.

Neat Stuff.

8. Billy is in the sixth grade, and he estimates that the books he carries to school weigh about 25 pounds. He figures he's right to within 3 pounds, give or take.

Billy's dog Woody weighs about 18 pounds, give or take 2 pounds.

(a) Billy says the books and the dog combined should weigh about 43 pounds, but how accurate could this be in "give or take" terms?
(b) One day, Billy decides to carry Woody in his school bag instead of his books. A lighter load! Give an estimate for the difference in the weights, in "give or take" terms.

It depends on how many Milk-Bones Woody's been eating lately.

9. *Make a game* that is about one-third likely to be won. Explain clearly how the game is played, and what the winning condition is. The best games are simple to play but complex in their potential outcomes. Don't worry too much about making the winning probability exactly one-third.

So, one game would be "Roll a die and if it comes up 1 or 2, you win." But you can make something more interesting!

10. Devise an experiment using one or more six-sided dice with exactly a one-*fifth* probability of success.

Ack! Stupid dice and their six sides.

11. With 240 coin flips come 120 pairs of flips.

(a) The four outcomes for a pair of flips are HH, HT, TH, TT. Given 120 pairs of flips, how many of each outcome would you expect?

(b) Find a 240-flip data set and determine the number of each paired outcome that occurred. Do you find any grounds to suspect that the data is fake?

12. What's the probability that an integer picked from 1 to n is a perfect square if

(a) $n = 10$?

(b) $n = 100$?

(c) $n = 1000$?

(d) $n = 10000$?

(e) What is happening "in the long run" (as n grows larger without bound)?

13. What's the probability that an integer picked from 1 to n is *square free* if

An integer is *square free* if it has no square factors greater than 1. So 4 isn't a factor, 9 isn't, and so on. 30 is square free; 60 isn't. You might need some technology help for this one.

(a) $n = 10$?

(b) $n = 100$?

(c) $n = 1000$?

(d) $n = 10000$?

(e) What is happening "in the long run" (as n grows larger without bound)?

14. Some of the terms in the Farey sequence have consecutive Fibonacci numbers as numerator and denominator. One such fraction is $\frac{5}{8}$, while another is $\frac{21}{34}$.

Where do these fractions appear within the Farey sequence? Can you find any justification?

15. Show that there must be a sequence of 100 consecutive non-prime positive integers.

16. On average, how far is it from one prime number to the next? How would you go about measuring such a thing?

Useless Stuff.

17. How many ways are there to use white and red Cuisenaire rods to build a cube, two units on a side? You are allowed to place red rods in any direction, including vertically.

This is a very nice problem, but we're not kidding: it's useless stuff.

Tough Stuff.

18. What's the *average* length of car used when you make all the trains of length 5? Is there a general rule at work here? Can you justify it?

19. Find a generalization to the problems in today's "Game of the Day" that can work for n people or prizes instead of 4. Specifically, what happens to the probability that everyone ends up in the wrong seat?

20. Henri and Tatyana play a very long game. They flip a coin: if it's heads, Henri gets a point. If it's tails, Tatyana gets a point. What makes it such a long game? Well, in order to win, you have to be 20 points ahead of your opponent.

How long, on average, will this game last?

21. A person is standing at the edge of a pool, and they've had one too many. Each step they take, they have a $\frac{1}{3}$ chance of stepping toward the pool, and a $\frac{2}{3}$ chance of stepping away from the pool.

Must be coming back from the Margarita-Off or something.

What is the exact probability that they eventually fall into the pool? Note that this probability will be more than $\frac{1}{3}$ since the first step could take them into the deep end.

THIS SPACE INTENTIONALLY LEFT BLANK

Well, not really, but we've already asked 21 questions and had extra space. Anybody want a spatula?

2007.6 For the Love of the Game

Game of the Day: "Shell Game"

In the Shell Game, a prize ball is placed under one of four shells. The player is asked four true-false questions: for each one they get right, they get to place a marker next to any of the four shells. If they place a marker next to the prize ball, they win.

Once, the player accidentally looked under the shell instead of placing the marker. The prize ball wasn't there. They placed the marker there anyway . . . brilliant!

1. (a) Assuming that the true-false questions are all answered with 50-50 likelihood, what is the probability that the player answers all four questions right (and automatically wins)?

(b) Find the probability that the player gets 0 right, 1 right, 2 right, 3 right, 4 right.

(c) Use expected value to find the probability that the player wins the Shell Game. For example, if the player gets 3 true-false questions right, they will *then* have a $\frac{3}{4}$ chance of winning the game.

Mmm, symmetry.

Important Stuff.

2. If you have not yet done problem 7 from yesterday, do so now. Either way, use the calculator to find `nCr(4,k)` for each k from 0 to 4; also find the value of $\binom{10}{3}$ and $\binom{10}{7}$.

Shouldn't we have said "values" here?

3. There are 16 trains of length 5. Lars, apparently impersonating Butch Cassidy, steals one of the trains at random. What is the probability that the train he stole was made from exactly 3 rods?

Or David Cassidy, we forget which. Come on, get happy!

4. Flip four coins. What is the probability of getting exactly two heads?

5. Herb stands at the corner of Center St and Main in beautiful downtown Logan, Utah. Assume for the sake of this problem that Logan is a grid of streets (pretty close, actually). Center goes east-west, while Main goes north-south.

Herb flips a coin. If he flips heads, he goes east one full block on Center. If he flips tails, he goes west one full block. Then he flips again.

(a) After two flips, where could Herb be and with what probability?

(b) After three flips, where could Herb be and with what probability?
(c) After four flips? Five?
(d) Find the probability that Herb is back at the corner of Center and Main after ten flips. You'll probably want to use that `nCr` thing for this.

6. Quick calculator skill today: to get a fraction to appear in the entry window of the calculator, hit **ctrl** (the blue button) then the division key. Then you can type whatever you want in the numerator and denominator.

Use the calculators to expand this expression:

$$\left(x + \frac{1}{x}\right)^4$$

Any thoughts? What about the 10th power?

The other templates were found with **ctrl** then the multiplication key. Look back to Session 4 for instructions on how to expand stuff; but keep in mind you can type out the word `expand` like a function.

7. What's the probability that a randomly chosen positive integer is

(a) divisible by 3?
(b) divisible by 5?
(c) not divisible by 3?
(d) not divisible by 5?
(e) not divisible by either 3 or 5?
(f) divisible by either 3 or 5?

8. Pick two positive integers at random, and call them m and n.

(a) What is the probability that m is even? What is the probability that *both* m and n are even?
(b) What is the probability that m and n do *not* share a common factor of 2? You can use the result from part (a) to solve this.
(c) What is the probability that m and n do *not* share a common factor of 3?
(d) What is the probability that m and n do *not* share a common factor of 5?
(e) What is the probability that m and n do *not* share any of the common factors 2, 3, or 5?

What's the chance that they *do* share the factor?

Neat Stuff.

9. This table gives the number of elements for the Farey sequence of order n. The third column gives the difference between consecutive terms.

The notation $|F_n|$ just means the number of elements in F_n. As an example of how Δ works, look at 19, 23, and the 4. Either $23 - 19 = 4$, or $19 + 4 = 23$. When you go from F_7 to F_8, four terms get added for a new total of 23. Hm, this table must be useful if it's taking up half a page . . .

n	$\lvert F_n \rvert$	Δ
1	2	1
2	3	2
3	5	2
4	7	4
5	11	2
6	13	6
7	19	4
8	23	6
9	29	4
10	33	
11		
12		
13		
14		
15		

Continue this table to F_{15} by considering how many elements get added to the Farey sequence each time.

So, what would this look like as a graph?

10. So, there are 16 trains of length 5 and 16 ways to flip 4 coins. Can you think of a way to match them up? It would be something like "this coin flip corresponds to this train".

This is usually called a *one-to-one correspondence* and is a typical solution method in counting problems.

11. Using your correspondence from problem 10, what train is represented by this sequence of heads and tails?

HHTTTTHTHHHHH

Note there may be more than one correct answer here since it depends on the correspondence used.

12. In yesterday's Cross-Program Activity, an interview process was mentioned. Candidates come in, and you must decide to accept them immediately or reject them forever. The goal is to have the best chance of finding the one best candidate among the group. One strategy is to reject the first n candidates out of hand, then accept the next candidate that is better than anyone seen so far.

Another application of this process is finding a good parking space: you might pass up the first few spaces you find expecting to see a better one later, or take a spot only to find a closer one you could've had. Could also apply to relationships . . .

Suppose there are 5 candidates to be seen.

(a) Oscar decids to accept the first candidate. What is the probability that Oscar gets the best possible candidate?

(b) Judy decides to reject the first candidate, then take the next one *that beats everyone so far*. What is the probability that Judy gets the best candidate this way?

(c) Same question, but Allen throws out the first two candidates before looking for the best of the rest.

With technology you might consider expanding the problem to larger groups . . .

13. On average, how far is it from one prime number to the next? How would you go about measuring such a thing?

14. Today's Game of the Day is a little simplified, since the player usually does better than 50-50 on the true-false questions in Shell Game.

(a) Suppose the player gets a true-false question right with probability $p = 0.75$. What is the chance that they win the Shell Game?

(b) Find a formula, in terms of p, for the probability that the player wins the Shell Game if they get questions right with probability p. Check that the formula gives the right answers for $p = 0, p = 0.5, p = 1$.

Tough Stuff.

15. Take the harmonic series and remove all the terms with the number 1 in their denominators:

Sadly this means the fraction $\frac{1}{231}$ is missing.

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{9} + \frac{1}{20} + \frac{1}{22} + \frac{1}{23} + \cdots$$

The normal harmonic series diverges (gets larger with no maximum). So, what about this one?

16. Take the harmonic series but consider only terms with prime denominators:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$$

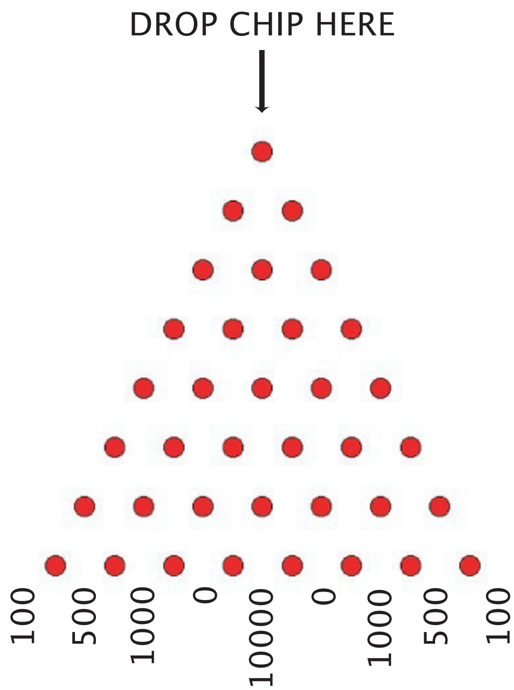
Does this sum converge? If so, to what? If not, can you prove it?

2007.7 Bored Games?

Game of the Day: Plinko!

Plinko is a favorite: all a player has to do is drop a chip, and they can win up to $10,000. Here's a graphic of a simplified version of the Plinko board:

Dude, it's Free Slurpee Day. Go get one. Or several.



Whenever the chip hits a Peg, it has a 50-50 chance of going left or right to the next one. At the bottom, it will fall into one of nine slots with dollar amounts on them, from $0 (d'oh) to $10,000 (woo hoo).

Poor Peg, continually getting beaned by Plinko chips.

1. (a) Describe the relationship between the falling Plinko chip and coin flipping.

(b) Find the probability of falling into the center slot for a $10,000 win.

(c) Find the probability of falling into a $0 slot. Careful: there are two of them.

(d) Find the probability of falling into $1000, $500, $100.

(e) How much, on average, will you win per chip if they let you keep playing this game for a long, long time?

(f) The "real" Plinko is typically played with 5 chips. How much, on average, should players expect to win if they have 5 chips to work with?

Important Stuff.

2. Pick a random integer between 1 and 21, inclusive. What is the probability that this number is . . .

(a) divisible by 3?
(b) divisible by 7?
(c) not divisible by 3?
(d) not divisible by 7?
(e) not divisible by 3, and also not divisible by 7?
(f) divisible by either 3 or 7?

3. Suppose we changed problem 1 to be numbers from 1 to 210, inclusive. Would anything change in the probabilities?

4. Pick any random positive integer. What is the probability that this number is . . .

Ben says, "This is the counting numbers." Bowen says, "Naturally."

(a) divisible by 3?
(b) divisible by 7?
(c) not divisible by 3?
(d) not divisible by 7?
(e) not divisible by 3, and also not divisible by 7?
(f) divisible by either 3 or 7?

What were we thinking, not using 7 and 11 on Free Slurpee Day?

5. **Calculator skill time.** Here's how to make a scatter plot for the Farey sequence data from yesterday. This one is longer! We will be using this again later in the week(s) ahead.

- Hit the HOME button then select option 5 to get a new document.
- When it asks you what kind of page you want, select Lists & Spreadsheet.
- In the first column type out the numbers 1 through 10.
- In the second column type out the number of elements in the Farey sequence of orders 1 through 10.
- Now go to the *top* of Column A to a small cell right next to the letter A, and type the word `order`. Do the same at the top of Column B and type the word `farey`. This step adds these columns as list data to be used in the other types of page.
- Take a breath for a sec as you flip to the next page.

Find this data in yesterday's Neat Stuff or your earlier notes.

- Hit the HOME button then select option 2 to get a new Graphs & Geometry page. Note that this should *not* be a new document.
- Hit the MENU button then select option 3, “Graph Type” and choice 3, “Scatter Plot.” (Hit enter.)
- Hit enter to bring up a list of x variables, and select “order”. Then hit tab. Do the same to select “farey” as the y variable. The scatter plot should appear.
- Hit the MENU button then select option 4, “Window” and choice 9, “Zoom - Stat”, to fit the window to your data.
- Hit the MENU botton then select option 5, “Trace”, to move along the data points.
- Celebrate with your favorite dance.

If you need to move between pages, hit the **ctrl** button then the left or right wheel arrow.

Are we there yet? Are we there yet?

6. Herb stands at the corner of Center St and Main in beautiful downtown Logan, Utah. Assume for the sake of this problem that Logan is a grid of streets (pretty close, actually). Center goes east-west, while Main goes north-south.

Herb flips a coin twice. The first flip tells Herb whether he’s going to move north-south or east-west. The second flip tells Herb which direction to go: heads for east or north; tails for west or south. All moves are one block.

Basically all four directions are equally likely with probability $\frac{1}{4}$. After moving a block, Herb decides where to go next.

(a) After two flips, where could Herb be and with what probability?

(b) After three flips, where could Herb be and with what probability?

(c) After four flips?

(d) Find the probability that Herb is back at the corner of Center and Main after six flips. This one is pretty hard without finding a pattern, so feel free to skip.

Poor Herb forgot to pack his tetrahedral dice or Shell Game cups.

7. Ben says that yesterday’s problem 6 included a wacky polynomial meant to describe Herb’s first walk. Do problem 6 if you haven’t. Can you find an even more wacky polynomial that models Herb’s second walk?

This comment was partially censored; Ben actually said . . . oh well. Ask him yourself.

Neat Stuff.

8. Why would they put the $10,000 space in the middle surrounded by two zeros, instead of the other way around?

Think mathematically!

9. Pick an integer between 1 and 900, inclusive. Find the probability that this integer is *square free*; that is, it has no square numbers greater than 1 as factors: no 4, no 9, no 16, and so on.

10. Pick two integers, each between 1 and 30, including duplications and order. There are 900 possible sets, including (21,17) and (17,21). Find the probability that the two numbers do not share a common factor greater than 1.

11. The number of Plinko chips a player actually plays with is 1, plus 4 more they can win at a pricing game. Typically players get 80% of the pricing game questions right. Find the expected payout the show pays for one game of Plinko.

12. Herb moves to Cubertville where the streets have no name. But they run in all directions: east, west, north, south, *up, down*. Herb rolls a die and moves a block; now there are six choices for where he's going to move. But the questions remain:

(a) After two flips, where could Herb be and with what probability?

(b) After three flips, where could Herb be and with what probability?

(c) Find the probability that Herb is back at the corner of Center, Main, and Pogo Blvd (the street that goes up and down) after five flips.

Alright, only these three streets have names.

Tough Stuff.

13. Take the harmonic series and remove all the terms with the number 2 in their denominators:

Sadly this means the fraction $\frac{1}{231}$ is missing.

$$1 + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{19} + \frac{1}{30} + \frac{1}{31} + \frac{1}{33} + \cdots$$

The normal harmonic series diverges (gets larger with no maximum). So, what about this one?

14. Take the harmonic series but consider only terms with prime denominators. Will that converge? If so, to what? If not, can you prove it?

2007.8 Shall We Play A Game?

Game of the Day: "The Big Wheel"

After three contestants on The Price is Right comes the Showcase Showdown, where the player who earns the most moves on. In this simplified version, each player spins twice on a wheel with the numbers 25, 50, 75, 100 on it. For this version, there is no "busting" by going over 100.

1. Find all the possible outcomes of spinning this wheel twice, and the probability that each occurs.

2. What is the average number of points Mary could expect to earn in one spin? two spins?

Mary gives it a good spin and is proud of how it keeps turnin'.

3. Expand this:

$$\left(x^{25} + x^{50} + x^{75} + x^{100}\right)^2$$

What's going on with the exponents and coefficients?

Important Stuff.

4. (a) Find the probability of flipping exactly 4 heads in 8 coin flips.

(b) Use your mad Plinko skillz to answer this question from Day 1 without guessing:

What is the probability of getting *exactly* 120 heads in 240 coin flips?

This is the probability of paying Amy $10,000 . . .

5. Let x and y be integers from 1 to 15, inclusive. There are 225 possible ordered pairs (x, y). For each (x, y), plot it if the greatest common factor of x and y is 1.

We have provided grids for these in the back if you need them. Need more? We got more. All part of our attempts to achieve plentiful sufficiency.

6. *On a separate grid* and *in a different color* plot all the (x, y) where the greatest common factor is 2. Now stare at the grids for a bit.

7. Pick a random integer between 1 and 52, inclusive. What is the probability that this number is . . .

(a) divisible by 4?
(b) divisible by 13?
(c) not divisible by 4?
(d) not divisible by 13?
(e) not divisible by 4, and also not divisible by 13?

Would this better be called indivisible? With liberty and justice for all?

(f) divisible by either 4 or 13?

8. **Calculator skill time.** This builds on yesterday's sketch, and shows how to plot a best-fit function to data, sometimes called a *regression*. Take a deep breath . . .

- Completely complete yesterday's problem 5. You should now have two pages: a spreadsheet page with the Farey sequence data, and a graphing page with the scatter plot.
- Go back to the spreadsheet page by hitting the **ctrl** button then the left wheel arrow.
- Position the cursor in cell D1 of the spreadsheet.
- Hit the MENU button then select option 4, "Statistics", then option 1, "Stat Calculations . . . ", then choose either option 6 or option A for a quadratic or exponential fit.
- A dialog box should come up. For "X List" choose `count`, then hit the TAB button. For "Y List" choose `farey`. Then hit enter twice and watch the magic. The spreadsheet page should now contain the coefficients of the fit in its columns, and the variable `f1` should now store the fit equation.
- Go back to the graphing page by hitting the **ctrl** button then the right wheel arrow.
- Hit the MENU button, then option 3, "Graph Type", then choice 1, "Function".
- Hit the up wheel arrow once to select function `f1` if it isn't already. You should see the regression equation here, perhaps with a few too many digits.
- Hit enter, and the fitting curve will display.

So which is better: a quadratic fit or an exponential one? How could you decide?

This is the godmother of all calculator skill tasks. If you're lucky, you picked up an nSpire with yesterday's task already finished.

Which fit to pick? You decide. Maybe do both.

One great thing about nSpire is that if you went back to add more data to the spreadsheet, everything *immediately updates* to reflect the changes. The list variables, the scatter plot, the fit coefficients, the regression plot... everything!

9. *Variance* is a measure of spread of data: a large variance indicates a wide spread, while a small variance indicates data is tightly packed. In this problem, we're going to show you how to calculate the variance of a data set. Variance is less often called the *mean squared deviation*, but that name describes how to calculate it:

- Find the *deviation* for each element, compared to the mean of the data.
- Take all the deviations and *square* them.
- Find the *mean* of all these values.

Spread 'em.

Consider the data $\{1, 7, 13, 25, 34\}$. The mean is... hm.

(a) What is the mean?

(b) Complete this table to find the variance of the data set. Some of the numbers have been filled in for you.

Data	Mean	Deviation	Square
1		-15	
7			
13			9
25			
34			324

Variance = **mean of squares** = $\cdots$

(c) The *standard deviation* of a data set is the square root of the variance. So, what is it for this set?

Psst: try doing this on the nSpire spreadsheet. The "Mean" column will contain the same number each time.

Standard deviation occurs when you teach something not on "the list".

Neat Stuff.

10. (a) Find the variance and standard deviation for the data $\{25, 50, 75, 100\}$ as seen in today's Game of the Day.

(b) Find the variance and standard deviation for the data set from *two* spins of the wheel. Note that duplicates *should* be included: there are 3 ways to make 100, and count them all.

(c) Notice anything interesting?

You calculated the mean for one and two spins in problem 2.

"Yes" is not a complete answer here.

11. The real "Big Wheel" has twenty numbers from 5 to 100, in increments of 5. Also, you bust if you go over 100 combined from two spins.

(a) Find a polynomial that could be used to represent the wheel. No, you don't have to write it all out.

(b) Hey, try this on the calculator:

$$\frac{x^{105} - x^5}{x^5 - 1}$$

(c) Find the probability that you bust if you spin twice, even including the idiotic move of spinning again if your first spin was 100.

Bah, I guess you might have to expand it too.

12. Pick a card from a deck of 52. What is the probability that this card is . . .

(a) a spade?

(b) an ace?

(c) not a spade?

(d) not an ace?

(e) not a spade, and also not an ace?
(f) either a spade or an ace?
(g) This remind you of anything?

13. Problem 7f is kind of interesting: the probability of an integer being divisible by either 4 or 13 is $\frac{4}{13}$. Does this ever happen again? If so, find the general case.

Tough Stuff.

14. So, the "Big Wheel" problem has a little more to it, in terms of strategy. When you spin the wheel the first time, you actually can stop with what you've got, or take a second spin. The goal is to be the closest to 100 without going over.

The tough part is reading this all the way through. There is a lot of cognitive demand here. But hey, it's Tough Stuff, deal with it.

But what's interesting is that only the last player has full information: they know exactly what amount they need to beat, so it's clear when they should stop or keep going. But the second player doesn't have this.

Suppose the first player busted out, and the second player's first spin comes up 50 (on the real wheel that goes from 5 to 100). Should they keep going, or stop? Neither option is really desirable, but which is better? Find a cutoff that says the second player should stop if they spin X or above, and keep going below that.

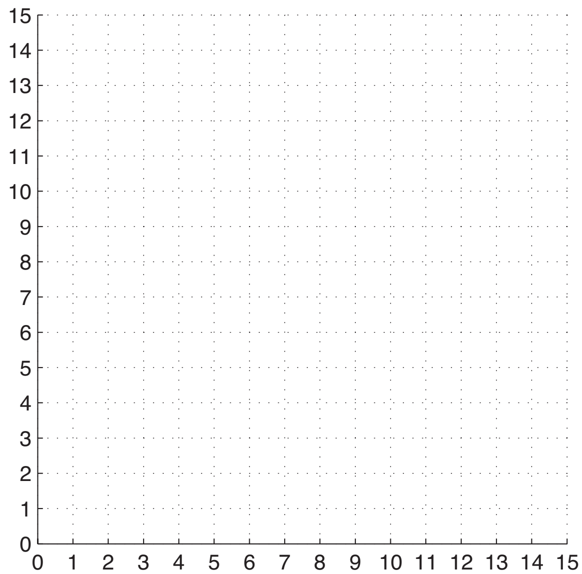
An even more challenging question is to analyze the first player's decision, since the later players could each beat them. Say the first player's first spin comes up 60. Should they keep going, or stop? If they stop they're pretty likely to get beat by someone, but if they keep going, they are more likely to bust than to improve. Find a cutoff that says the first player should stop if they spin X or above, and keep going below that.

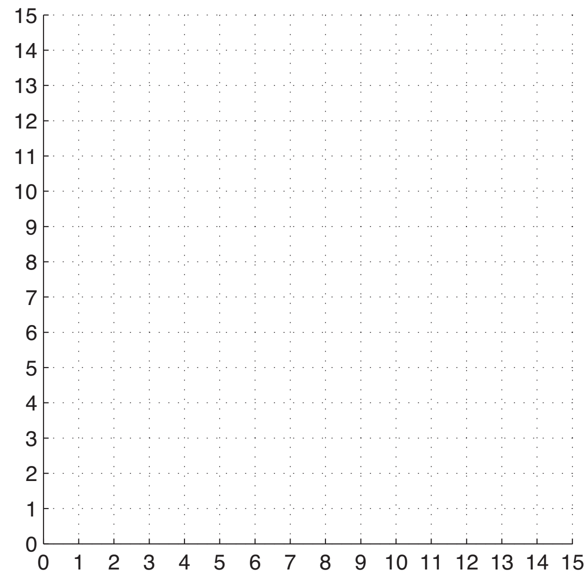
15. There is a continuous analog to the "Big Wheel.", where any real number between 0 and 100 is equally likely to be "spun". The rules remain the same: go over 100 and you're out, and only the best score is a winner.

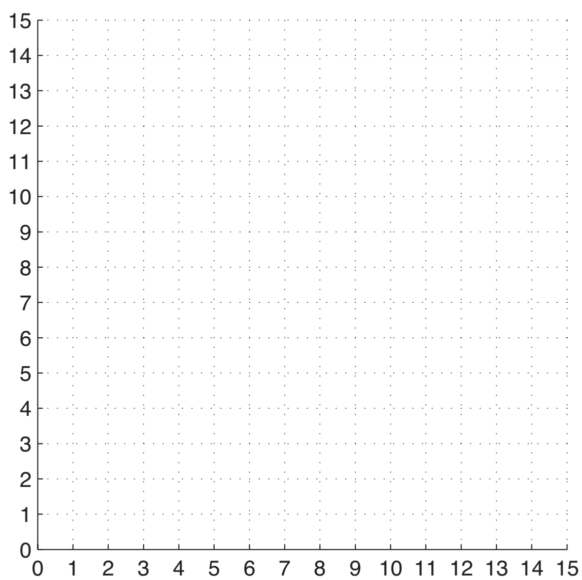
Develop strategies for the second and first players to determine their first spin cutoffs. You will need some calculus techniques to attack this problem.

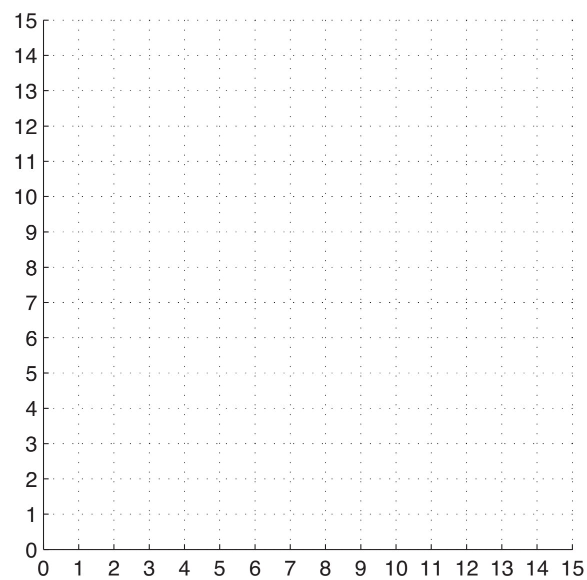
15-by-15 grids

Graph paper also a good idea.









2007.9 Big Game Hunter

Game of the Day: "Golden Road"

Golden Road is the richest game available on The Price is Right. Players start given (for free!) a small prize, and try to win bigger and bigger prizes.

Consider a game of Golden Road with three possible outcomes:

If your actual Golden Road game ended with only a $6,000 prize, you'd be ticked. Most large Golden Road prizes are over $50,000.

- One-fifth of the time, the player wins a lousy spatula worth $5.
- Two-fifths of the time, the player wins a lovely designer rug worth $500.
- Two-fifths of the time, the player wins the big prize, a trip to Banff worth $6,000.

1. Find the average payout for this game in the long run.

2. *Variance* is a measure of spread of data: a large variance indicates a wide spread, while a small variance indicates data is tightly packed. Variance is less often called the *mean squared deviation*, but that name describes how to calculate it:

As a spread, you might say "I can't believe it's not variance!" But really it is.

- Find the *deviation* for each element, compared to the mean of the data. Deviation can be positive or negative.
- Take all the deviations and *square* them.
- Find the *mean* of all these values.

So, let's calculate the variance for one play of "Golden Road".

(a) Complete this table to find the variance for one play. Some of the numbers have been filled in for you. The "Mean" column will contain the value you found in problem 1.

Psst: try doing this on the nSpire spreadsheet. The "Mean" column will contain the same number each time.

Data	Mean	Deviation	Square
5		-2596	
500			
500			4414201
6000			
6000			

Variance = **mean of squares** = $\cdots$

(b) Wow, that variance sure seems big. The *standard deviation* of a data set is the square root of the variance. So, what is the standard deviation for one play of "Golden Road"?

Standard deviation occurs when you teach something not on "the list".

Important Stuff.

3. Peter plays a game with five outcomes, but only remembers a few things about its probabilities. One of the five outcomes is the most likely of all five, so he calls its probability p. Two of the four other outcomes have probability two-thirds of p, and the remaining two outcomes are even less likely, only one-sixth of p.

Peter wants to calculate p. What is it? He's listed all five outcomes.

It's unclear whether Peter named the variable after himself.

If you're not sure where to start, try a value for p and see how it goes. Eventually, set up an equation to solve for p.

4. Work with your tablemates to fill out four transparencies, each of which is a 15-by-15 grid.

- Using the **blue** pen on the first transparency, plot all ordered pairs (x, y) with $1 \le x, y \le 15$ such that x and y have no common factors greater than 1. (This was problem 5 on yesterday's set.)
- Using the **red** pen on the second transparency, plot all ordered pairs (x, y) with $1 \le x, y \le 15$ such that the greatest common factor between x and y is exactly 2.
- Using the **green** pen on the third transparency, plot all ordered pairs (x, y) with $1 \le x, y \le 15$ such that the greatest common factor between x and y is exactly 3.
- Using the **black** pen on the fourth transparency, plot all ordered pairs (x, y) with $1 \le x, y \le 15$ such that the greatest common factor between x and y is exactly 4.

5. Now pile all the transparencies together one atop another. What do you notice? Can you explain this?

6. Count the number of dots on the blue and red transparencies. Roughly how many times more blue dots are there than red dots?

7. This data comes from a 240-by-240 grid instead of a 15-by-15 grid. It gives more information about long-term trends.

Color	Dots (out of 57,600)
blue	35,087
red	8,771
green	3,931
black	2,203

We thought about having you fill out the 240-by-240 grid and count it up, but it's only a two-hour class.

(a) Roughly how many times more blue dots are there than red dots?
(b) Roughly how many times more blue dots are there than green dots?
(c) Roughly how many times more blue dots are there than black dots?

8. Check out the zone on the blue transparency with x and y between 1 and 7. See it anywhere else? Is anything like this going on for other colors? Can you use this to explain the patterns in problem 7?

Check, check, check . . . check it out!

They say if you stare at these long enough, you'll see a sailboat or a dolphin.

9. Pick any point (x, y) in a *very* large grid. Let the probability that (x, y) is a blue dot be p.

(a) Give an approximate value for p based on the data in exercise 7.
(b) If the probability of picking a blue dot is p, what is the probability of picking a red dot?
(c) If the probability of picking a blue dot is p, what is the probability of picking a green dot?
(d) Suppose this went on forever: what is the probability, compared to p, that the pair (x, y) will have greatest common factor n?

10. Let p be as in exercise 9. Explain why

$$p + \frac{1}{4}p + \frac{1}{9}p + \frac{1}{16}p + \cdots = 1$$

then use this to write an expression for the value of p.

Neat Stuff.

11. Calculate each of these to six decimal places:

(a)

$$\frac{1}{\sum_{n=1}^{5} \frac{1}{n^2}}$$

(b)

$$\frac{1}{\sum_{n=1}^{25} \frac{1}{n^2}}$$

(c)

$$\frac{1}{\sum_{n=1}^{100} \frac{1}{n^2}}$$

These are the results from problem 10, cut off after 5 terms, 25 terms, and so on. Hit the blue **ctrl** button before entering if you'd like an approximate answer instead of an exact one.

Apologies for a slight technical glitch. Each denominator should look more like

$$\sum_{n=1}^{5} \frac{1}{n^2}$$

(d)

$$\frac{1}{\sum_{n=1}^{1000} \frac{1}{n^2}}$$

(e)

$$\frac{1}{\sum_{n=1}^{10000} \frac{1}{n^2}}$$

What do you notice about the accuracy here?

12. You remember Herb's one-dimensional walk, east-west along Center St? Of course you do! If not, go back and work it out.

(a) Find the *variance* and *standard deviation* for how far away from the start Herb is after two steps.

(b) Find the *variance* and *standard deviation* for how far away from the start Herb is after three steps. Include all eight possible ways.

(c) Repeat for four steps. Wow, that works out nicely for four steps, doesn't it?

(d) What happens in general?

After two steps, Herb is either 0 or 2 blocks from the start.

13. Repeat problem 12 for the two-dimensional case where Herb goes north, south, east, or west. What happens now? Consider Herb's distance from his starting point in Euclidean terms—so, after two steps, he is either 0, 2, or $\sqrt{2}$ blocks from where he started.

This is a pretty amazing result if you ask me. But I guess you didn't! Still, pretty cool.

14. What about Plinko from the top of a tetrahedron with 6 rows? How would this work, and what numbers would come from the Pascalization? What numbers appear at the bottom of the tetrahedron? What numbers appear on the lateral faces?

15. Expand this using the nSpire:

$$(a + b + c)^6$$

What's up with that?

Bonus problem. If train A leaves Chicago at 50 mph, and train B leaves Denver at 60 mph, what is the probability that you'll get this problem right?

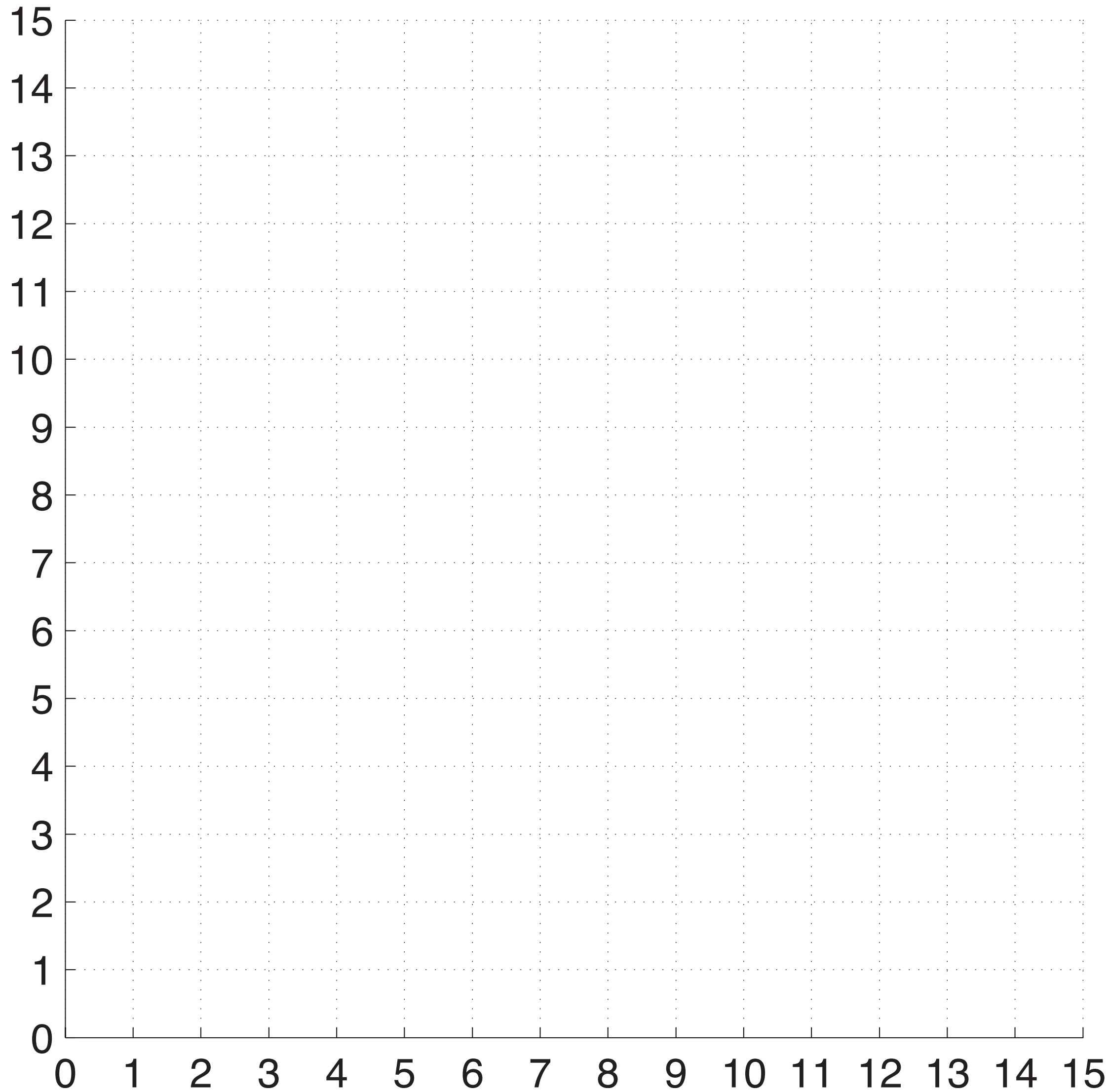
Tough Stuff.

16. Repeat problem 12 for the three-dimensional case where Herb goes north, south, east, west, up, or down. What happens now? Again, consider Herb's distance from his starting point in Euclidean terms—after three steps, he could be 1, 3, $\sqrt{5}$, or $\sqrt{3}$ units from the start.

15-by-15 grid

Please use these transparencies, not graph paper. You'll see something interesting! Use blue for 1, red for 2, green for 3, black for 4.

1,2,3,4, do this or we'll sing some more!



2007.10 This Tastes Game-y

Game of the Day: "M&M Survivor"

Gather up 40 M&Ms together in a cup and toss them on a plate. Some will be M-side up, count these and return them to the cup. Eat the others.

Repeat a total of six times, and see how many survive. Survivors ready . . . go!

i can has m&m?

If using other things such as pennies, ignore this last step.

1. Complete this table with the number of survivors from your experiment:

# Tosses	Survivors
0	40
1	
2	
3	
4	
5	
6	

We decided it would not be appropriate to play this with people, what with the eating and all. Plus if you weren't wearing a shirt with an M on it, you're in real trouble.

2. What type of fitting curve might you expect for this data? If time permits, use technology to find and display the fitting curve.

Would Gloria Gaynor enjoy this game?

Important Stuff.

3. (a) Pick two positive integers at random. What, approximately, is the probability that the two integers do not share a common factor greater than 1?

Using positive integers feels natural.

(b) We looked at this equation on Friday:

$$p + \frac{1}{4}p + \frac{1}{9}p + \frac{1}{16}p + \cdots = 1$$

What's this p stand for?

This has no affiliation to the Sue Grafton book "P is for Peril." Yes, we know how to use the internets!

4. Evaluate this on the nSpire:

$$\frac{1}{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}}$$

Say what? There's an infinity button?? Use **ctrl** and the weird book thing on the right.

Holy moly.

This page has set the record for Most Wacky Sidenotes.

5. Calculator skill time. Your calculator should be loaded with a document called "farey50", but if it isn't, here's what you do.

I can only imagine the graphing calculator games of this new generation. It will put "Drug Wars" to shame.

- Go to the top-level menu by continually hitting the **ctrl** button and the up arrow. This should take you to a list of documents.
- Find another calculator with "farey50" on it. This document is located in the "Examples" directory.
- Get a link cable. Attach the side marked A to the calculator you are transferring *from*, and the side marked B... well, yeah.
- On the calculator you are transferring *from*, highlight the "farey50" document.
- Select the TOOLS menu by hitting the **ctrl** button and hitting the HOME button in the very top right.
- Select option 1, "File", then option 5, "Send".
- Watch the magic as you now have "farey50" in the Examples directory on your own calculator.

Don't let your godmother catch you doing this.

Okay, everybody got it? Now open the "farey50" document within the Examples directory by clicking on it. The first page is a spreadsheet with the data and the coefficients for a best-fit quadratic. The second page is a scatter plot of the data with the best-fit quadratic overlaid. Wow, that's close!

Hit **ctrl** and up to get to the menu, then open the Examples directory if necessary. If it asks you to save the current document, make like Nancy Reagan.

In any case, decide how you would estimate the number of elements in F_{100}, the Farey sequence of order 100, and for F_{200}. Once your group comes up with answers, we'll let you know how close they are.

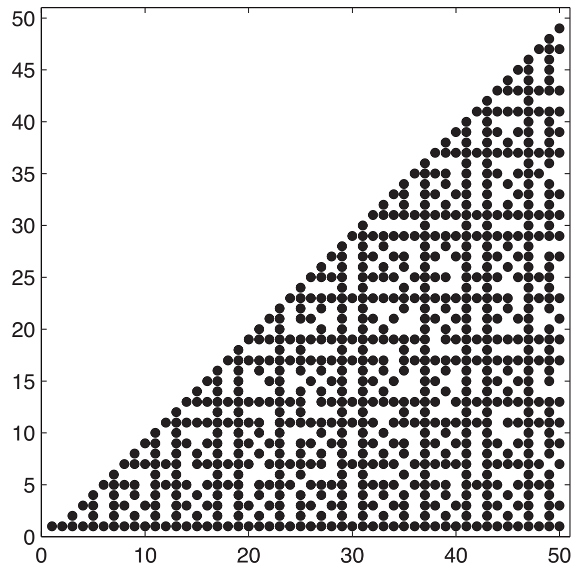
6. The best-fit quadratic for the first 50 Farey data points seems to be

$$f(x) = .3056x^2 + .2388x + 1.608$$

(a) As x gets larger, what terms become more important or less important in this quadratic?

(b) At the top of the next page is a plot of the points in F_{50}: if $\frac{y}{x}$ is in the Farey sequence, then (x, y) is plotted. Give an estimate for the probability that if I pick one of the 2500 points in the square $1 \le x, y \le 50$, it is in the Farey sequence.

Maybe a picture of F_{15} would be more helpful?



Neat Stuff.

7. Here's an interesting sequence of, uh, sequences.

Index	Sequence
1	1, 1
2	1, 2, 1
3	1, 3, 2, 3, 1
4	1, 4, 3, 2, 3, 4, 1
5	1, 5, 4, 3, 5, 2, 5, 3, 4, 5, 1

Each sequence *inserts* the index number n in all places where two consecutive terms in the sequence above add to n. The 5s are placed between 1 and 4, then 3 and 2, then 2 and 3, then 4 and 1. Anywhere it adds to 5.

(a) Build the next three sequences. In the first, insert 6s anywhere you see consecutive terms adding to 6.

(b) How many elements are in each sequence? Wowzers.

The heck, there was supposed to be a 17.

8. Look back at the lists of Farey sequences from earlier in the course.

For some students, this is their most favoritest fraction problem evah!

(a) What is the first fraction placed between $\frac{2}{5}$ and $\frac{1}{2}$?

(b) What is the first fraction placed between $\frac{3}{7}$ and $\frac{1}{3}$?

(c) What is the first fraction placed between $\frac{2}{5}$ and $\frac{3}{7}$?

(d) What is the first fraction placed between $\frac{3}{5}$ and $\frac{5}{8}$?

(e) What is the first fraction placed between $\frac{a}{c}$ and $\frac{b}{d}$?

9. Suppose you were standing at the origin $(0, 0)$ and there were 2500 points of light, one at every point (x, y) with integer coordinates 50 or less. You pan across from the east to the north, and in between you see a whole lot of points. You can't see any point blocked by another one: for example, you can't see $(10, 6)$ because $(5, 3)$ is in the way.

(a) Approximately what percentage of all 2500 points do you see?
(b) What are the very first points you see?
(c) What is the first point you see that *doesn't* have y-coordinate 1?
(d) What point do you see exactly halfway along this panning? By this we mean in terms of points seen, not in terms of angle.
(e) What point is halfway through the "first half" of the panning? Can you explain why? You might want to consider some simpler cases first. Again, this is in terms of points seen, not the point at a 22.5° angle.

Sheesh, a Reagan and Bush reference on the same problem set? What have we come to? Still it's better than the "Over and Ova-Reagan" reference from 2005.

10. Find the *variance* and *standard deviation* for the number of heads tossed when rolling

(a) ... four dice.
(b) ... five dice.
(c) ... nine dice.
(d) ... n dice.

11. You spin a wheel with the numbers 25, 50, 75, 100 on it. Find the *variance* and *standard deviation* for the total score from spinning the wheel

(a) ... once.
(b) ... twice.
(c) ... four times.
(d) ... 100 times. (Guess?)

Go on, spin it! You know you want to. Just be sure to make appropriate "boop boop boop" noises as it goes around.

12. Find the probability that a positive integer chosen at random is *square free*; that is, it has no factors greater than 1 that are perfect squares.

Tough Stuff.

13. Pick three positive integers. Find the probability that they do not *all* share a common factor greater than 1.

For example, the set $\{15, 21, 35\}$ do *not* share a common factor.

2007.11 Oh, The Games You'll Play!

Game of the Day: "Wheel of Fish"

On the classic show "Wheel of Fish", players spin a wheel and earn a number of fish based on their spin. The four stops on the wheel are

one fish

two fish

three red fish

ten blue fish

They can then exchange their fish for the contents of a mystery box, but that's really beside the point. We realize you know how to play, but we're going to invite contestants up here just for the halibut.

1. Find the mean, variance, and standard deviation for the number of fish earned from one spin of this wheel.

2. Find the mean, variance, and standard deviation for the number of fish earned from two spins of this wheel, with the 16 possible outcomes

$$2, 3, 4, 11, 3, 4, 5, 12, 4, 5, 6, 13, 11, 12, 13, 20$$

Avery points out that 2^4 and 4^2 are the same. Wow! Impressive. If only it worked for 5.

3. Expand this on the nSpire:

$$(f + f^2 + f^3 + f^{10})^3$$

Find the probability that you earn exactly 6 fish in three spins.

Important Stuff.

4. (a) How many numbers less than or equal to 15 do *not* share a common factor with 15?

(b) How many numbers less than or equal to 35 do *not* share a common factor with 35?

(c) How many numbers less than or equal to 91 do *not* share a common factor with 91?

(d) Multiply this out:

$$\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

In our "Dead Giveaway" department, the numbers for 4(a) are: 1, 2, 4, 7, 8, 11, 13, 14. Oh and by "common factor" we mean "not 1".

5. Consider the numbers 1 through 35.

(a) What fraction of these numbers are divisible by 5? by 7?

(b) What fraction of these numbers are *not* divisible by 5? by 7?

(c) Write down the numbers 1 to 35. Cross out any number that is divisible by 5. What fraction of the original 35 numbers remain?

(d) Now cross out any number that is divisible by 7. What fraction of the numbers that survived part (c) also survived this second cut?

(e) What fraction of the original 35 numbers survived both cuts?

Usually people stop at 10 or 20 or 100, so the numbers 1 through 35 rarely get proper consideration. Not much to say about 35 really. It's one more than that miracle street, and one less than a square triangular number. Poor guy. At least 35 is the highest number you can reach counting on your fingers in base 6.

6. So there's this function ϕ that takes positive integers as input. You take the input and find all its prime factors. For each prime factor, you multiply through by $(1 - \frac{1}{p})$. For example, take 15. The primes are 3 and 5, so the result is

$$\phi(15) = 15\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

(a) Calculate $\phi(15), \phi(35)$, and $\phi(91)$.

(b) Calculate $\phi(105)$ and $\phi(231)$.

(c) Calculate $\phi(9)$ and $\phi(27)$. Watch out!

(d) Calculate $\phi(8675309)$.

(e) Describe what ϕ measures in your own words.

Naturally it would. This function is pronounced "fee" of n, which you might hear if a giant approaches an Englishman.

The nSpire can factor numbers if you need it to. It also makes Julienne fries.

7. Complete this table for $\phi(n)$ along with the cumulative total of all ϕ values from 1 to n.

n	$\phi(n)$	$\sum \phi(n)$
1	1	1
2	1	2
3	2	4
4		
5		
6		12
7		
8		
9		
10		32

How many elements are in F_8, the Farey sequence of order 8? What about F_9? Hm.

8. Describe and explain some connections between the ϕ function and the Farey sequences.

Neat Stuff.

9. Look back at the lists of Farey sequences from earlier in the course.

Kids like adding fractions this way. But then again, they also like paste and Pokemon.

(a) What is the first fraction placed between $\frac{2}{5}$ and $\frac{1}{2}$?
(b) What is the first fraction placed between $\frac{2}{7}$ and $\frac{1}{3}$?
(c) What is the first fraction placed between $\frac{2}{5}$ and $\frac{3}{7}$?
(d) What is the first fraction placed between $\frac{3}{5}$ and $\frac{5}{8}$?
(e) Prove that if $\frac{a}{c} < \frac{b}{d}$ for positive a, b, c, d, then

$$\frac{a}{c} < \frac{a+b}{c+d} < \frac{b}{d}$$

10. Suppose you were standing at the origin $(0, 0)$ and there were 900 points of light, one at every point (x, y) with integer coordinates 30 or less. You pan across from the east to the north, and in between you see a whole lot of points. You can't see any point blocked by another one: for example, you can't see $(24, 21)$ because $(8, 7)$ is in the way.

We refuse to do a Madonna joke.

(a) Approximately what percentage of all 900 points do you see?
(b) What are the very first points you see?
(c) What is the first point you see that *doesn't* have y-coordinate 1?
(d) What point do you see exactly halfway along this panning? By this we mean in terms of points seen, not in terms of angle.
(e) What point is halfway through the "first half" of the panning? Can you explain why? No trig is needed.

At this point, Sendhil might sing, "Whoa, we're halfway there . . . "

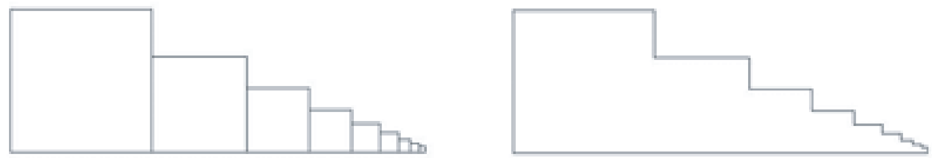
At this point, Sendhil might sing, "Whoa, we're approximately one-fourth of the way there . . . "

11. (a) You spin the Wheel of Fish four times. What's the most likely number of fish you'll win? Note that this is not the same as the mean, or even the median.
(b) What's the most likely number of fish from *ten* spins of the wheel?

Guh! What a mess! And I'm just talking about the big pile of fish.

11. Cathy imagines an "infinite stairway", created by lining up an infinite number of squares. The first square is 1 by 1, the second is $\frac{1}{2}$ by $\frac{1}{2}$, the third is $\frac{1}{3}$ by $\frac{1}{3}$, and the nth is $\frac{1}{n}$ by $\frac{1}{n}$

And she's buying a stairway to 11. Or, maybe this would be better as a Spinal Tap reference?



So, what's the perimeter and area of this infinite staircase?

Tough Stuff.

12. Build a histogram for the number of ways (or the probability, take your pick) you can get each possible outcome from 10 spins of the Wheel of Fish. For example, there are 49,905 ways to spin the wheel 10 times and earn exactly 36 fish.

A bar graph is *unacceptable*!

13. Can $\phi(n)$ ever be less than $\frac{n}{10}$? Explain.

14. So now you know (perhaps from the calculator) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

What about the sum of reciprocals of *odd* squares only? That is,

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$$

or in summation form

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Try figuring this out without the use of technology.

15. So $2^4 = 4^2$, and that's interesting: it's the only pair of positive integers x and y with $x^y = y^x$ and $x \neq y$. When else does $x^y = y^x$ when $x \neq y$ for positive reals x and y? Never? Ever after? Graph?

What a cool slash for not equal. Is that the same slash as the one in GnR?

2007.12 The Crying Game

Game of the Day: "Who Wants To Be A Millionaire?"

In 2000 it was on four times a week, and it was routinely the #1 through #4 shows in the ratings. On the show, multiple-choice questions with four options are presented.

Question 1 was usually really easy, but it didn't stop someone from saying that Hannibal crossed the Alps on llamas. *Oops!*

Picture yourself on Millionaire and totally spooked by the cameras. You guess at each question. Ignore (for now) the fact that if you get a question wrong, you're off the show.

Well, you got the first question wrong . . . here, have another chance!

1. You get asked two multiple-choice questions and take a guess at each. Find the mean and variance for the *number of questions* you will get right. There are 16 possible outcomes here, and they are

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2$$

Are you sick of this particular fraction yet? It sure comes up often.

2. Same for three questions: what is the mean and variance for the number of questions you will get right?

3. Expand the polynomial $(r + 3w)^4$ and use it to find the mean and variance for the number of questions you get right in four tries. Or use your own method.

Say, this would be a pretty good \$1 million question.

Some Millionaire-style math problems are available at www.ams.org/wwtbam/archive.

Important Stuff.

4. Pick two positive integers at random. What is the probability that they do not share a common factor greater than 1? Give your answer to six decimal places.

Here's a table of the 36 ordered pairs of numbers (x, y) with $1 \le x, y \le 6$.

Looks like what you could get from rolling two dice to me.

$(1,6)$	$(2,6)$	$(3,6)$	$(4,6)$	$(5,6)$	$(6,6)$
$(1,5)$	$(2,5)$	$(3,5)$	$(4,5)$	$(5,5)$	$(6,5)$
$(1,4)$	$(2,4)$	$(3,4)$	$(4,4)$	$(5,4)$	$(6,4)$
$(1,3)$	$(2,3)$	$(3,3)$	$(4,3)$	$(5,3)$	$(6,3)$
$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$	$(6,2)$
$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(6,1)$

5. Look for a pattern to these answers.

(a) What fraction of these ordered pairs have *both* numbers divisible by 2?

How can there be a pattern when there's only one question?

(b) What fraction of these ordered pairs do *not* have both numbers divisible by 2?

(c) What fraction of these ordered pairs have *both* numbers divisible by 3?

(d) What fraction of these ordered pairs do *not* have both numbers divisible by 3?

(e) Cross out any of the 36 ordered pairs where both numbers are divisible by 2. What fraction of the original 36 ordered pairs remain?

(f) Now cross out any remaining ordered pair where both numbers are divisible by 3. What fraction of the numbers that survived part (e) also survived this second cut?

(g) What fraction of the original 36 ordered pairs survived both cuts?

Oh. Two pages.

This is the obligatory reference to Christopher Cross and/or the unrelated duo Kris Kross. This problem is totally krossed out!

6. Build a grid with the numbers 1 through 15 as labels on the sides. You now have 225 ordered pairs.

(a) How many of the 225 ordered pairs do *not* have 3 as a common factor (in both numbers)? For example, $(9, 6)$ has 3 as a common factor, but $(10, 12)$ does not.

(b) How many of the 225 ordered pairs do *not* have 5 as a common factor (in both numbers)?

(c) How many of the 225 ordered pairs do *not* have either 3 or 5 as a common factor (in both numbers)?

(d) If Gloria picks an ordered pair at random, what is the probability that it does *not* have either 3 or 5 as a common factor (in both numbers)?

(e) Multiply this out:

$$\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)$$

7. Calculator skill time. Using a calculator page, it is possible to define custom functions.

- Get a new calculator page by hitting HOME, option 5 (New Document), then selecting Calculator.
- Hit MENU, then select option 1 “Tools”, then option 1, “Define”.
- Type: $f(n) = 1 - \frac{1}{n^2}$. Your calculator line should look like

$$\text{Define } f(n) = 1 - \frac{1}{n^2}$$

You could also type out the word d-e-f-i-n-e, then a space (next to Z). Nah.

- Hit enter, and watch no magic as the calculator just says "Done".
- Now type $f(3)$ and hit enter. Hey hey! I feel a sense of deja vu.

It's either deja vu or something I ate.

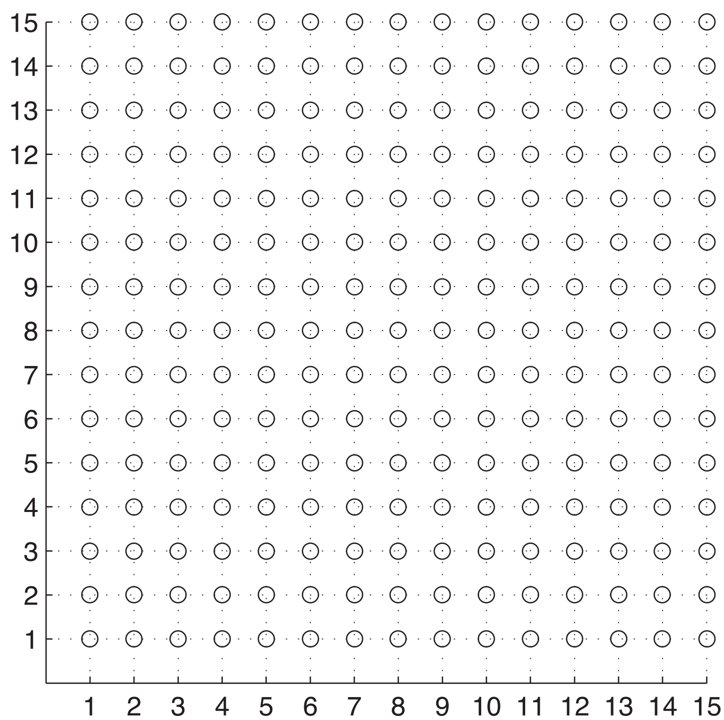
Find this product using your new function:

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)$$

and give your answer as a decimal using the blue **ctrl** button.

Will you still need me, will you still feed me... when I'm...

8. Here is a coordinate plane, 15-by-15, with an open circle at each point (x, y) with integer coordinates.



(a) Using a red pen, color in the circles for any point where both x and y share a common factor of 2.

(b) Using the same pen, color in any additional circles for any point where both x and y share a common factor of 3.

(c) Using the same pen, color in any additional circles for any point where both x and y share a common factor of 4. Oh, okay. Well, that was easy. Next page!

What, you ate your red pen? Fine, do what you like! But use the same easily seen color.

(d) Using the same pen, color in any additional circles where both x and y share a common factor of 5.
(e) Do it again for 7; for 11; for 13. Oh my.

Neat Stuff.

9. Calculate this product as long as necessary:

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)$$

The giant π just means multiply, so this is the same exact thing as

Mmmm, giant pie. Man, there's a lot of food references today. By the way, it's sigma for sum and pi for product, if that helps to remember them.

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\cdots$$

where each denominator is a prime. Say, I think you have a calculator function to simplify this work. When you multiply all these "slightly less than 1" things together, what happens?

This is a pretty interesting result, though, wicked awesome as they say. Take enough terms until you're happy. The primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, and maybe a few more.

10. A parallelogram has vertices $(0, 0)$, (c, a), (d, b), and $(c + d, a + b)$. Find its area in terms of the variables given. Also, find the area of a few parallelograms formed by consecutive "Farey fractions". Take your pick.

This is not a subtle reference to Pick's Theorem. But, that would've been a good idea.

11. Ah, back to Millionaire. Let's say that you're not totally terrible at answering the questions. You can answer any of the first 5 questions with 90% probability, any of the middle 5 questions with 75% probability, and any of the last 5 questions with 60% probability. Find the mean and variance for the number of questions you get right *before missing one*. Sorry, no lifelines!

Bookkeeping can be tricky here, so feel free to approximate a bit.

12. What's the probability that a positive integer is *square free*? That is, 4 is not a factor, and 9 is not a factor, and 16 is not a factor.... I guess we can skip 16 because of 4.

Tough Stuff.

13. Use Pick's Theorem to show that if $\frac{a}{c}$ and $\frac{b}{d}$ are consecutive "Farey fractions", then $bc - ad = 1$ every time.

Ask around about Pick's Theorem or look it up. It's a great Geoboard theorem.

14. Show that if $\frac{a}{c}$ and $\frac{b}{d}$ are consecutive "Farey fractions", then the fraction with least denominator between $\frac{a}{c}$ and $\frac{b}{d}$ is $\frac{a+b}{c+d}$.

2007.13 End Game

Game of the Day: "Press Your Luck"

On "Press Your Luck" players take turns at a board filled with fabulous cash and prizes. But they'll have to avoid the whammy. On the show, the whammy killed off all your money. In this version, the whammy is worth −\$5000. The other possible outcomes are 3000, 6000, 7000, 9000, and 10000, all equally likely. Actually we'll look at the data without the thousands.

Check out these dice! We're really being constructivist now.

Rumor has it that the whammy killed the radio star, too.

1. Find the mean, variance, and standard deviation for one "spin" (or roll). The data set is

$$3, 6, 7, 9, 10, -5$$

2. Find the mean, variance, and standard deviation for two "spins" (or rolls). There are now 36 possible outcomes. Divide the labor if you like.

Union rules prohibit us from telling you how to break up the labor. Intersection rules, too.

3. Fill in the missing pieces here to build an expression you could use to find the distribution for three "spins".

$$(x \quad + x \quad + x^7 + x \quad + x \quad + x \quad)$$

By the way, the mean score for 3 spins is 15 and the variance is 75. (So what's the standard deviation, then?)

4. Use the results from the first three problems to predict the mean, variance, and standard deviation for four spins, 25 spins, 100 spins.

Having to do this 100 times by hand would be a dicey situation.

Important Stuff.

5. (a) You spin the Wheel of Fish once. What is the mean and variance for what comes? (The possible outcomes were 1, 2, 3, 10.)

(b) You roll a standard die once. What is the mean and variance for what comes? (The possible outcomes are 1, 2, 3, 4, 5, 6.)

(c) In the Wheel of Fish bonus round, the player spins the wheel, then also gets to roll a die. They earn the total number of fish from both results. What is the mean and variance for these 24 possible outcomes? The table at the top of the next page might prove useful.

Please, no more fish!!! I thought we were done with the fish!!!

	1	2	3	4	5	6
1						
2						
3						
10						

6. Expand this product:

$$(x + x^2 + x^3 + x^{10})(x + x^2 + x^3 + x^4 + x^5 + x^6)$$

Any thoughts?

Please only include thoughts about this math problem.

7. The actual Wheel of Fish bonus round has the player roll 12 dice at once, then spin the Wheel of Fish six times. The total number earned from all of this is the amount the player will take home, in pounds of fish. Use the properties of mean and variance to find the mean, variance, and standard deviation for how much fish a player stands to win in the bonus round.

¡*No mas pescados*!

8. Flip a coin. If it comes up heads, record a 1. If it comes up tails, record a 0.

(a) If you flip the coin once, what is the mean, variance, and standard deviation for the total count (aka the number of heads)?
(b) *Two* coin flips?
(c) *Three* coin flips?
(d) *Twenty-five* coin flips?
(e) *240* coin flips?
(f) n coin flips?

Hm, 240? That sounds familiar somehow. And yet so long ago.

9. Answer a multiple-choice question with four options without looking. If you were right, record a 1. If you were wrong, record a 0.

(a) If you answer one question, what is the mean, variance, and standard deviation for the total count (the number of questions answered correctly)?
(b) *Two* questions?
(c) *Three* questions?
(d) *Four* questions?
(e) *48* questions?
(f) n questions?

Most of the answers can be found in yesterday's work on the Millionaire game.

Neat Stuff.

10. You run an experiment that has probability p of success. The probability of failure is $1 - p$.

(a) Find the mean and variance for performing the experiment once, in terms of p.

(b) *Two* runs of the experiment?

(c) n runs?

(d) Many statistics books include the rule for standard deviation for "Bernoulli trials" which are experiments with probability of success p and failure q. Explain how you know the rule for the standard deviation σ is

$$\sigma = \sqrt{npq}$$

The last two problems describe two such experiments: rolling a die ($p = .5$) and answering a trivia question ($p = .25$).

Apparently the stat books want you to mind your p's and q's.

11. (a) Find the probability that if you guess at 3 multiple choice questions, you will get exactly 2 of them right. Each question has four possible answers.

(b) Expand this and figure out what is up with that:

$$(0.25r + 0.75w)^3$$

12. Wheel of Fish runs a "Tournament of Champions" where all fish values are doubled: the wheel has 2, 4, 6, and 20. What happens to the mean, variance, and standard deviation for the number of fish in one spin? (They were 4, $\frac{25}{2}$, and $\frac{5}{\sqrt{2}}$ in the original game.)

To qualify for the Tournament of Champions, you must perform the Macarena while eating a fish.

13. Wheel of Fish once ran a Celebrity Week where all fish values were 5 more than normal: the wheel had 6, 7, 8, and 15. What happened to the mean, variance, and standard deviation then?

For the love of Pete, please, please, NO MORE FISH. Celebrity week on "Wheel of Fish" featured Todd Bridges, Florence Stanley, and Abe Vigoda. What, you don't get it?

14. Spin two spinners, and look at the sum of what comes. The first spinner is your generic 1-2-3 spinner with three equal zones. The second spinner has only two zomes, labeled 0 and 3.

(a) Find the mean and standard deviation for the sum you get from the two spinners.

(b) Multiply this out:

$$(x + x^2 + x^3)(1 + x^3)$$

Any thoughts?

As featured in the upcoming Tom Clancy novel "The Sum of All Spinners".

Review Your Stuff.

The final day of this course is mostly taken up by review problems. So we think it would be a good idea for groups to form some summarizing questions that come out of whatever you might find valuable in this course. So, we want your *table* to write *two problems* on any subject that has cropped up in the course.

Here are some topics you might consider writing problems about.

Farey sequences	Common factors or lack thereof
Pascal's triangle	Expected value
Variance	Standard deviation
Counting with polynomials	General probability

The goal is to create a review whose problems get at the fact that we've come a long, long way in three weeks. The problems should help others synthesize their learning of the aforementioned topics.

Ooh, "aforementioned". That's like a three-dollar word.

So, don't write any stumpers; consider yourself writing two problems that could both fit into "Important Stuff." If your table wants to write more than two, that's fine, and the extra questions can be a little more "Neat" or "Tough" or even "Game"-y. We reserve the right to pile your problem(s) into a category called "Random Stuff"; no offense is intended. And it's okay to be funny, as long as it doesn't get in the way of the math.

So if your table wants to submit a game suggestion for the final day, go ahead, but you're still on the hook for two straight-away review problems. No tricks plz!

Tough Stuff.

15. Let $z = x + (\sqrt{1-x^2})i$ be a complex number with $|x| < 1$. Use the nSpire to help here unless you really want some tough algebra.

(a) What's the magnitude of z?
(b) Expand z^2 and look at the real part. Hm? Zeros?
(c) Expand z^3 and look at the real part. Hmmm? Zeros? Connection to some earlier question, perhaps?
(d) Expand z^4 and look at the real part. Hmmmmm?
(e) Graph the real part of z^2, z^3, z^4 as a function of x.
(f) What might z^n look like?

Pafnuty says hi. Who's that? Look it up! And you thought you'd get through the whole course without hearing from him.

Bonus Tracks

This section is reserved for a few things we weren't able to get to in the course. Have a look, play around if you like.

16. It's possible to make two different, distinct dice that *don't* have 1 through 6 on them, but their sums behave the same as the sums from regular dice. The rules for these dice are

- All numbers on each face must be positive integers.
- When the two dice are combined, there is 1 way to get a sum of 2, 2 ways to get a sum of 3, ... 6 ways to get a sum of 7, ... 1 way to get a sum of 12.

What's the only other way to make dice do this? Can factoring help you?

17. Can you make 3 distinct dice that behave like 3 regular dice would? What about 4 dice?

18. Prime numbers seem to get rarer and rarer as the numbers get higher.

(a) How many primes are there between 1 and 100?
(b) ... between 101 and 200?
(c) ... between 201 and 300?
(d) Make a long table of prime numbers, along with the "gap" between them. For example, there is a gap of 6 between 23 and 29.

19. Show that there is no limit to the size of the "gap" between consecutive primes.

20. The gaps between primes seem to not be very well-behaved. But, neither are the primes. One way of trying to smooth this is to look at the *average gap* between primes. For example, the average gap among the first 5 primes is

$$\frac{1+2+2+4}{4} = \frac{9}{4}$$

The first 5 primes are 2, 3, 5, 7, 11; the four gaps are 1, 2, 2, 4.

Calculate the average gap for

(a) the first 10 primes
(b) the first 50 primes (the 50th is 227, starring Jackee)
(c) the first 250 primes (the 250th is 1579)
(d) the first 1250 primes (the 1250th is 10,177)

Any thoughts?

21. The function $p(n)$ takes in n and returns the nth prime.

(a) Show that $\frac{p(n)}{n}$ gives the probability that a positive integer less than or equal to n is prime.
(b) What happens to this function as n gets larger? Does it approach a limit... or...?

22. The function $p(n)$ takes in n and returns the nth prime number. Consider another function that takes in this same n and returns the *average gap* among the first n prime numbers. (You were asked to calculate this function for several values of n.) Get some more data points and investigate this function. What does this investigation suggest about the prime numbers?

23. How does a calculator find the "line of best fit"?

24. Show that the line of best fit always passes through the balance point of a data set; that is, the point $(\overline{x}, \overline{y})$ that would be the center of mass or centroid of the data.

25. Recently we calculated this product:

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)$$

What happens if you remove the square and try to calculate...

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)$$

Use this to prove that there an infinite number of primes.

26. Meanwhile, why stop at squares? Investigate

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)$$

for various choices of the parameter s. (We called it s for a reason, by the way.)

2007.14 Game Over, Man!

Game of the Day: "Deuce"

John and Jim are playing tennis. They reach "deuce", a point in the game where the next two points can determine a winner:

There is no limit to the number of times the game reaches "deuce" again. The most deuces ever in a pro tennis game is 37.

- If John wins both points, he wins the game.
- If Jim wins both points, he wins the game.
- If the two points are split, the game returns to "deuce" and continues.

John is serving, so he wins each point with probability 0.6. Find the probability that John wins the game before Jim does.

Important Stuff.

We have grouped the problems roughly by category. There is a purpose to the numbering. Flip around!

13. Flip 4 coins and roll 2 dice. What is the probability of getting exactly 2 tails *and* a sum of 7 on the dice?

8. (a) Build a table for the 36 outcomes for rolling two standard dice.
(b) Find the mean and variance for the two dice rolls.
(c) Expand $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$. What do you notice?
(d) Find the mean, variance, and standard deviation for rolling four dice.

You found the variance for rolling one die on Day 13. Die, another day.

10. Given what you know now, how would you approach the question of determining if a series of 240 coin flips came from a fair coin or was faked?

8. You are working your way through the lunch line in the PCMI tent. It is sandwich day, and there are five stations where you may reach across the table or stay on your side for your preferred option (you pick one of each dressing, pasta, bread, condiment, and meat).

(a) What is the probability that you will get each of your preferences at all five stations if you stay on your side and *never* have to reach? You'd need some good luck . . .
(b) What is the probability that you have to reach to get your preference *every time*?
(c) Expand $(a + b)^5$. What's up with that?

The probability that Mitch gets his preferred dressing is zero, because they don't ever have bleu cheese!

Yes, this table actually wrote "What's up with that?" in their problem. Nice one.

(d) Use the expansion to find the probability that you have to reach *3 or more* times to get your preference.

9. In a semi-popular variant on an old game, Bear mauls Ninja, Ninja sneaks up on Cowboy, and Cowboy shoots Bear.

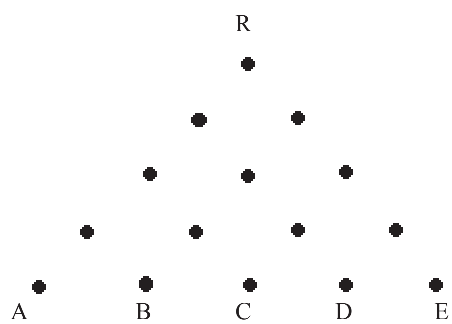
Avery just keeps picking ninja. He does it every time. Sendhil never wises up on this, though, and picks Bear and Cowboy with equal probability. Find the probability that

(a) Avery wins three games in a row
(b) Avery wins exactly two out of three consecutive games
(c) Avery wins a best-of-five tournament by winning 3, 4, or 5 games out of 5

Note that tables from opposite sides of the room came up with these last two problems. Best-of-five tournament? Five stations? Hm.

8. Ralph makes like a Plinko chip and percolates in the random walk during the Fourth of July Parade. If Ralph starts at point R and can move downward either to the left or right each time to the next available row, what is the probability that Ralph ends up at point C?

In this process, Ralph runs into several people, not just Pegs.



6. Start at the origin and flip a coin. If it comes up heads move 1 unit to the right and 1 unit up. If it comes up tails, move 1 unit to the right and one unit down. Repeat this for 3 more flips (4 flips total).

(a) How many different paths end at the point (4,2)?
(b) Do any paths pass through (2,1)? Why or why not?

10. Find the fraction with smallest possible denominator between $\frac{3}{5}$ and $\frac{2}{3}$.

2. To supplement funding, PCMI holds a cow-pie derby. A 50-by-50 grid is marked and squares are sold. Since tickets

Cow pies are not square, nor do they easily divide by 6.

are sold only to mathematicians, they buy only squares with coordinates that have no common factors greater than 1. What is the probability that PCMI will *not* need to pay a prize? (Assume all possible tickets are sold, and that only one cow-pie is dropped.)

10. Pick two integers with $x \le y$ and $1 \le x, y \le 15$.

(a) Find the probability that the fraction $\frac{x}{y}$ is already written in lowest terms.

(b) Suppose we allow $x > y$ as well. What happens now?

Consider $\frac{9}{7}$ to be in lowest terms for the purposes of this problem, even though it is an improper (sometimes called "vulgar") fraction. Proper!

12. (a) For each element $\frac{y}{x}$ of F_6, the Farey sequence of order 6, plot the point (x, y).

(b) Find all points that intersect the line $y = x$.

(c) Find all points that intersect the line $y = x - 1$.

(d) Find all points that intersect the line $y = \frac{1}{2}x$.

(e) Find all points that intersect the line $y = 2x$.

11. (a) Imagine a golden spatula. Now ignore that and draw the graph of all numbers with $1 \le x, y \le 6$ with $x > y$ such that x and y have no common factors.

(b) Do you have all the elements of F_6? Explain. Why does this work?

(c) Think about it: what has your spatula learned?

If this doesn't make sense, don't blame us!

13. How many *more* fractions does the Farey sequence F_{210} have than the Farey sequence F_{209}?

A lovely problem from clap clap clap clap deep in the heart of Texas.

12. For each Farey sequence F_3 through F_6, find the mean, variance, and standard deviation of the numbers in the sequence. (For F_3 this is $0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$.) Any patterns or thoughts?

"You stink" does not qualify as a reasonable "thought" on this or any problem.

4. Angie is playing with a single suit of cards. Each card gets a value. Kings, queens, jacks, and tens are worth ten each. All other number cards (2 through 9) are worth 5. The ace is worth -15. So, don't draw the ace.

(a) Find the mean, variance, and standard deviation for 1 pick.

(b) Find the mean, variance, and standard deviation for 2 picks, drawing with replacement.

(c) Find the mean, variance, and standard deviation for 65 picks, drawing with replacement.

(d) Write an expression you could expand to help with this situation.

Shouldn't 64 picks work out more nicely? Apparently not!

11. Three people are picked for the Big Lotto Spin. The wheel contains 3, 5, 7, 10, and 15 million. Each contestant get to spin and win three times!

(a) Find the mean, variance, and standard deviation for one spin.

(b) Find the mean, variance, and standard deviation for two spins; for all three spins.

(c) How much money, on average, will the show give out in one episode (with three players spinning three times each)?

This puts "Who Wants To Be A Millionaire" to shame. Minimum prize 9 million! Where do I sign up?

12. Using the nSpire, find the mean, variance, and standard deviation for this data set:

$$2, 7, 8, 9, 10, 15, 20, 22$$

11. The marginally popular spin-off show Wheel of Lures is played with a five-stop wheel. Its values are $-1, 3, 5, 7, 11$.

(a) Find the mean, variance, and standard deviation for one spin of the Wheel.

(b) The game also uses an eight-sided die with the sides $-3, 4, 5, 6, 7, 9, 12, 16$. Find the same stats for the die.

(c) In the Final Countdown round, the wheel is spun twice and the die is rolled once. Find the variance for this round.

(d) Write an expression that, if expanded, would list all 200 outcomes for the Final Countdown round.

We changed the 8 to a 7. Hope you don't mind. Oh, and we changed the 10 to a 9 in the second set. Positive integers are our friends, naturally.

Sing along! *It's the final countdown!*

1. On the new game show "Prime Time," the lucky contestant spins a wheel with seven equal spaces, numbered 2, 3, 5, 7, 11, 13, and 17, which determines their winnings.

(a) If you randomly select two numbers from the wheel, what's the probability they have no common factor greater than 1?

(b) Find the mean, variance, and standard deviation of their winnings on one spin, two spins, three spins.

Prime Time is sponsored by Seventeen magazine, hence the top value 17. The note from table 1 says: "Bowen has never appeared in Seventeen Magazine. He was, however, in Tiger Beat after his pinball championship last year." Bowen replies: not true. It was Bikini Magazine.

1. After their sponsorship contract ends, the producers decide that there's not enough conflict in the show and start recruiting failed contestants from reality shows and rename the show "What's My Number Line?" They decide to change the numbers on the wheel as well, to $-7, -6, -4, -2, 2, 4$, and 8.

(a) What do you notice about these numbers compared to the old ones (2, 3, 5, 7, 11, 13, 17)?
(b) Find the mean, variance, and standard deviation of their winnings on one spin of this new wheel.
(c) How do your answers compare to the previous problem? Any conjectures?

6. In Mr. Bolognese's class, the students can choose 4 numbers out of 1-10 to make a Homework Lottery Ticket.

(a) How many possible Homework Lottery Tickets are there? Order doesn't matter. Use Pascal's Triangle to help.
(b) Chris then picks four numbers from 1 through 10 and awards homework passes for each matching number. Find the mean number of passes a student can expect to earn from this game.

Gambling is illegal at Bushwood. I hope the parents don't find out about this program!

9. Phil and Deb are lost in a game of D&D (Dungeons and Dragons). The game uses five dice, one for each Platonic solid: 4 sides, 6 sides, 8 sides, 12 sides, 20 sides. Each die is labeled from 1 to its total number of faces (like normal, so the cube is 1-6). Phil rolls all five dice.

(a) For each die, calculate the mean and variance for the number rolled. You might want to break this up among a group!
(b) Find the mean, variance, and standard deviation for the sum of the results on all five dice.

I don't know why Phil didn't just roll 2d20 and cast a Spell of Remembrance.

7. Use all of the following concepts in a cohesive mathematical narrative.

- $\sum \frac{1}{n^2}$
- integers with no square factors
- $\frac{3}{\pi^2}$
- points in the first quadrant visble from the origin
- $\frac{\pi^2}{6}$
- fractions in the Farey sequence visble from the origin
- $\frac{6}{\pi^2}$

* shrug * I guess read it and move on? It's funny at least.

Neat Stuff.

10. (a) Wheel A has the following outcomes: $1, 3, 5, 7, 9, 11, 13$. What is the mean, variance and standard deviation for the outcomes of one spin?

(b) Wheel B has the following outcomes: $0, 2, 2, 8$. What is the mean, variance and standard deviation for the outcomes of one spin?

(c) The object of a game is to spin both wheels and subtract the value of wheel B from wheel A. The contestant either earns or pays (if the result is negative) the amount. What is the mean, variance and standard deviation for the outcomes of one round?

(d) What might the product of two polynomials look like that could be used to find the outcomes of this game?

(e) Suppose 5 rounds are played. What is the expected amount of money earned in the 5 rounds? Use the result from 1 round to help.

5. There are two red fish and three blue fish in a fish tank. Tired of thinking about fish, you randomly pull out a fish one at a time and flush it down the toilet. You only stop after "eliminating" both of the red fish.

So, is this a Poisson distribution, or a distribution of poisson?

(a) What is the probability that you will stop after flushing the third fish?

(b) How many fish do you expect to flush overall?

(c) Generalize to r red fish and b blue fish.

5. Nicole and Darryl are the finalists in the International National Cowboy/Ninja/Bear Championship. A player wins the championship after winning 3 showdowns. What is the expected number of showdowns before a winner is determined? Remember, ties can occur.

For the rules, see earlier in the problem set. Like the Olympics, this event is held only every four years. Or, it seems, every night around these parts.

5. Jim and Bree play the following coin game. Jim is given the sequence of heads and tails THH. Bree is given the sequence HHH. The winner is the player whose sequence appears first. For example, if the sequence of coin flips were HHTHTTHH, Jim would win.

Jim must've set up these rules . . .

(a) What is the probability of Bree winning this game?

(b) How would the game change if Jim started with THH and Bree started with HTH?

13. In the sequence of lights with a 30-by-30 grid, what would be the first 5 lights *blocked* by other lights, panning from east to north?

2. PCMI participants receive grades for their performance on the problem sets. The grade comes from 3 options:

- Roll a die with six grades: A, B, C, C, D, F.
- Start with 75, then toss a coin. Heads gives 80 for a B, tails gives 70 for a C.
- Start with 50, and spin the Wheel of Fish five times. The number of fish is added to your grade.

Which of these gives the best strategy? (Does it depend on what you're after?)

Apparently, by asking for ¡No mas pescados! we were really saying "Only fish! Give us just fish! We love fish! Yes!" So says Table 9.

4. Bag 1 contains 3 green balls and 2 yellow balls. Bag 2 contains 2 green balls and 5 yellow balls. A ball is drawn at random from Bag 1, then placed into Bag 2. What is the probability of now drawing a green ball from Bag 2?

And after all this, who's left holding the bag?

Tough Stuff.

8. (a) Give a precise definition for the totient function $\phi(n)$.
(b) Calculate $\phi(4)$, $\phi(9)$, and $\phi(36)$.
(c) If a and b are positive integers, under what conditions does $\phi(a) \cdot \phi(b) = \phi(ab)$?

ϕ'd me, Seymour!

7. From the origin, look down the x-axis and slowly turn counterclockwise. We explored earlier where you are facing when you have see half of the total visible points. What fraction have you seen when your gaze is described by $y = nx$ for a natural numbers n? Use some of the colored grids you have made, then try to generalize.

7. For curious calculus afficionados. There is a connection between the derivative of a generating polynomial and the mean. From yesterday's "Press Your Luck" game,

$$\frac{d}{dx}\frac{(x^3 + x^6 + x^7 + x^9 + x^{10} + x^{-5})^2}{6^2}$$

gives ten when setting $x = 1$. Ten is the mean of rolling two Press Your Luck dice.

(a) Explain why this works.
(b) Find a connection variance and the second derivative.

0. Explain why $e^{\pi\sqrt{163}}$ is an integer. Amazing!

0. Suppose you were allowed to continue rolling the "Press Your Luck" die with sides 3, 6, 7, 9, 10, and one Whammy side, earning points until you either decided to quit or until you hit a Whammy. This time, the Whammy behaves like it should and takes *all* your earnings. So, when should you be willing to quit the game?