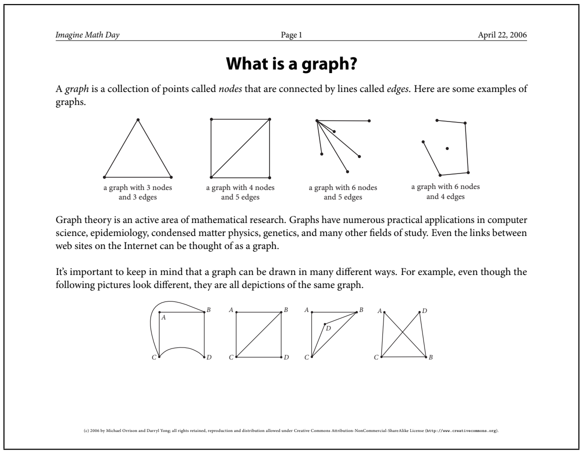
Instructor's Guide to the Graph Complexity Project

The goal of this activity is to encourage students to create and explore their own mathematics, to think and reason logically, to communicate their ideas to their peers, and to develop a joy for discovery in mathematics. Accordingly, it is essential that instructors emphasize that there are many ways to define graph complexity and that while some ideas may be more compelling than others, any definition that is mathematically correct can be a working definition.



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What is a graph?

A *graph* is a collection of points called *nodes* that are connected by lines called *edges*. Here are some examples of graphs.

a graph with 3 nodes and 3 edges
a graph with 4 nodes and 5 edges
a graph with 6 nodes and 5 edges
a graph with 6 nodes and 4 edges

Graph theory is an active area of mathematical research. Graphs have numerous practical applications in computer science, epidemiology, condensed matter physics, genetics, and many other fields of study. Even the links between web sites on the Internet can be thought of as a graph.

It's important to keep in mind that a graph can be drawn in many different ways. For example, even though the following pictures look different, they are all depictions of the same graph.

A B C D

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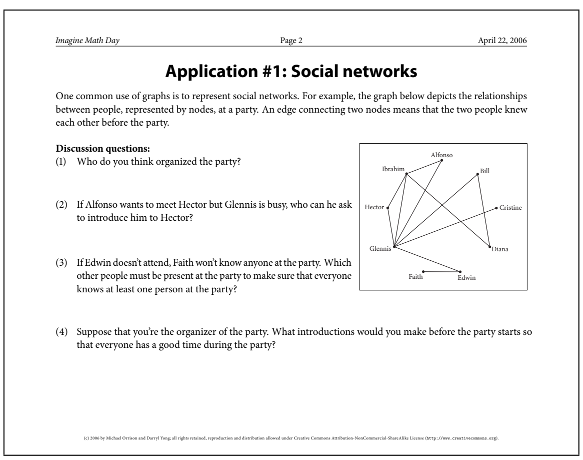
The terms "vertex" and "node" are used interchangeably in graph theory, as are the terms "edge" and "arc." We have chosen the word "node" because students often mistake the singular form of "vertices" for "vertice" (and pronounce it "verticy").

Even though the definition of a graph given on this page is quite general, all of the examples of graphs here are simple, non-directed graphs. A graph is allowed to have more than one edge connecting a pair of nodes, but that would make it non-simple. We have chosen to limit this already very rich graph complexity exploration to simple, unweighted, non-directed graphs. Students may wish to explore the complexity of these other types of graphs after this activity. Notice that a graph can have a node that is not connected to other nodes by an edge.

There are some areas of graph theory that are concerned about the way that the graph is drawn on a page. For example, the pins and copper connections on a printed circuit board form a graph whose edges cannot cross. (A graph is *planar* if it can be drawn in a plane without any edges crossing. A fun game based on this idea can be found at `http://planarity.net`.) However, it remains that the arrangement of the nodes and edges on the page are not part of a graph's make up.



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Application #1: Social networks

One common use of graphs is to represent social networks. For example, the graph below depicts the relationships between people, represented by nodes, at a party. An edge connecting two nodes means that the two people knew each other before the party.

Discussion questions:

(1) Who do you think organized the party?

(2) If Alfonso wants to meet Hector but Glennis is busy, who can he ask to introduce him to Hector?

(3) If Edwin doesn't attend, Faith won't know anyone at the party. Which other people must be present at the party to make sure that everyone knows at least one person at the party?

(4) Suppose that you're the organizer of the party. What introductions would you make before the party starts so that everyone has a good time during the party?

Alfonso
Ibrahim
Bill
Hector
Cristine
Glennis
Diana
Faith
Edwin

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The three application examples are designed to help students gain facility with graphs quickly and to introduce them to some common applications of graphs. Each application focuses on a different aspect of graph theory and suggests different ways of measuring graph complexity. The discussion questions are arranged so that they become increasingly open-ended. These open-ended questions have been intentionally written to allow for multiple interpretations, so that students can develop confidence in expressing their own opinions and in thinking creatively.

This first example of a graph as a social network encourages students to think about node degrees (number of edges attached to each node). The idea of measuring the sample variance of node degrees could arise in the context of this example.

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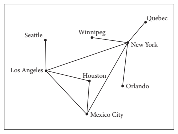
Application #2: Airline routes

The graph below is a simplified representation of the routes for BestFares Airlines. Each edge represents a regularly scheduled flight between two cities.

Discussion questions:

(5) Since there is no direct flight from Los Angeles to Quebec, travelers between these two cities must make a connection through what city?



(6) How many different ways are there to get from New York to Houston without revisiting a city?

(7) If BestFares Airlines is considering adding another flight among these cities, which two cities you would recommend connecting? Why?

(8) Suppose your boss has asked you, a widget salesperson, to travel to all of the cities shown above (in no particular order) to visit clients. Can you find an itinerary that minimizes the total number of flights?

This airline route example encourages students to think about walks and paths of graphs. The idea of measuring the girth of a graph may arise here. Question 8 is the classic "traveling salesman problem," one of the most intensely studied problems in computational mathematics and theoretical computer science. Students may enjoy thinking about where the optimal location of the corporate headquarters (beginning and end of the sales circuit) might be.

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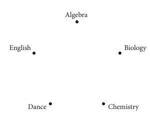
Application #3: Exam schedules

Here is a list of students and the final exams they must take. Suppose that you are in charge of scheduling the exams so that two exams aren't held at the same time if there is a student taking both exams.

Last Name	Exams
Morton	Algebra, Biology, Dance
Nguyen	Chemistry, English
Ort	Algebra, Biology
Pramanik	Chemistry
Quincy	Dance, English
Repa	Biology, Dance
Sakamoto	Biology, Dance, English
Torres	Algebra, Chemistry, English



Discussion Questions:

(9) Create a graph in which each node represents a subject. Draw an edge between two subjects if there is a student taking an exam for both subjects. Suppose that there are only two available exam time slots, 8 a.m. and 10 a.m. Is it possible to put each of the five exams in one of the two time slots without creating a conflict? How can you answer this question using the graph you've created?

(10) Suppose there are three available time slots: 8 a.m., 10 a.m., and noon. Can you schedule the exams to avoid conflicts? What is the minimum number of exam time slots needed so that there are no scheduling conflicts?

(11) How might the scheduling change if the chemistry exam was cancelled? What if the dance exam was cancelled?

Hidden in this example is the concept of the chromatic number of a graph, that is, the minimum number of colors required for every node to be colored so that every edge connects vertices of different colors. The time slots take the place of colors in this example.

Students wanting to explore this example further might enjoy thinking about this question: "If there are only three available time slots for the exams, how would you schedule the exams to create the least conflict?"

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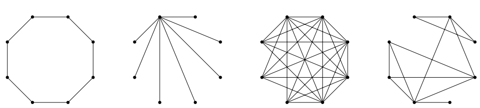
How complex is a graph?

Complexity is in the eye of the beholder.



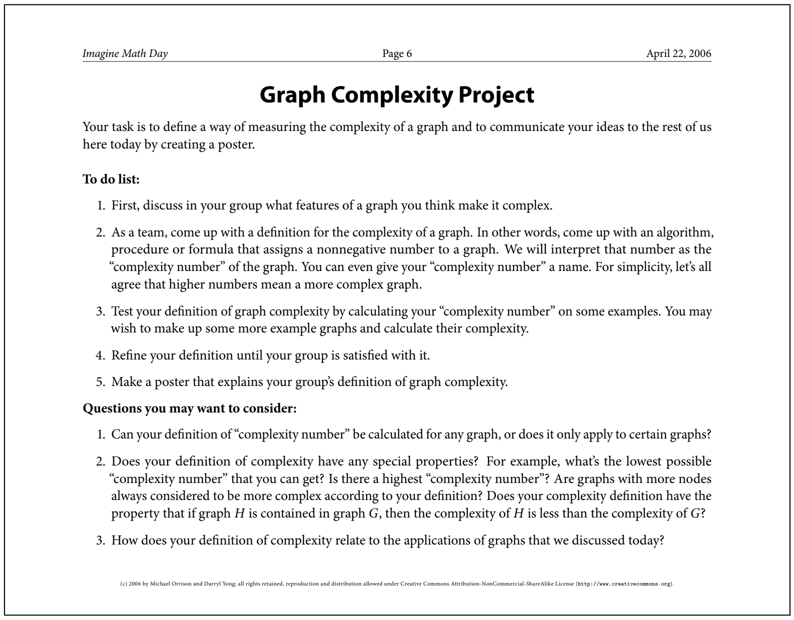
Which of the graphs above would you say are complex?

Can you articulate why you think one graph is more complex than another?

Can you rank these graphs from most complex to least complex?

To develop your ideas, you might try putting each of these graphs in the contexts of the applications on the previous pages.

Notice that all of the example graphs on this page were chosen to have eight nodes to encourage students to think of measures of graph complexity that involve more than simply counting the number of nodes and/or edges.



Graph Complexity Project

Your task is to define a way of measuring the complexity of a graph and to communicate your ideas to the rest of us here today by creating a poster.

To do list:

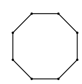
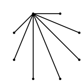
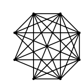
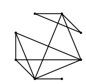
1. First, discuss in your group what features of a graph you think make it complex.
2. As a team, come up with a definition for the complexity of a graph. In other words, come up with an algorithm, procedure or formula that assigns a nonnegative number to a graph. We will interpret that number as the "complexity number" of the graph. You can even give your "complexity number" a name. For simplicity, let's all agree that higher numbers mean a more complex graph.
3. Test your definition of graph complexity by calculating your "complexity number" on some examples. You may wish to make up some more example graphs and calculate their complexity.
4. Refine your definition until your group is satisfied with it.
5. Make a poster that explains your group's definition of graph complexity.

Questions you may want to consider:

1. Can your definition of "complexity number" be calculated for any graph, or does it only apply to certain graphs?
2. Does your definition of complexity have any special properties? For example, what's the lowest possible "complexity number" that you can get? Is there a highest "complexity number"? Are graphs with more nodes always considered to be more complex according to your definition? Does your complexity definition have the property that if graph H is contained in graph G, then the complexity of H is less than the complexity of G?
3. How does your definition of complexity relate to the applications of graphs that we discussed today?

The culmination and heart of the activity.

Here are some possible definitions of graph complexity and how the four example graphs compare on these measures.

				
Girth	8	∞	3	3
Diameter	4	2	2	4
Chromatic number	2	2	6	4
Sample variance of node degrees (s^2_{N-1})	0	$\frac{9}{2}$	$\frac{15}{14}$	$\frac{12}{7}$
Number of distinct cycles	1	0	5	4
Average distance between two nodes	$\frac{16}{7}$	$\frac{7}{4}$	$\frac{31}{28}$	$\frac{25}{14}$
Minimum number of node removals required to create a disconnected graph	2	1	5	1

This activity is very open-ended and students can continue their explorations in many ways. Students can look at different measures of graph complexity and compare them with each other; they can also delve deeper into their own measures of graph complexity by looking for patterns or properties, then forming, testing and proving conjectures. In this activity, we have only considered simple, unweighted, non-directed graphs. Measures of graph complexity can also apply to other types of graphs as well.

Active research is being conducted in this area, and students may be enjoy learning that some of their definitions are actually used by researchers. Here are some other ways of measuring graph complexity that have appeared in the mathematics literature:

- the number of spanning trees of a graph [2],
- the value of a certain formula involving the number of nodes, edges, and proper paths in a graph [3],
- the minimum number of additions, subtractions, and scalar multiplications required to multiply an adjacency matrix of a graph and an arbitrary vector [4], and
- the number of Boolean operations, based on a predetermined set of Boolean operators (usually union and intersection) necessary to construct a graph from a fixed generating set of graphs [5].

Links to more information about graph theory:

- `http://en.wikipedia.org/wiki/Graph_theory`
- `http://mathworld.wolfram.com/Graph.html`
- `http://www.aisee.com/gallery`

Acknowledgments: The authors thank Jack Bloom, Peg Cagle, Oscar Chavarria, Pam Mason, Sue Mussack, Jerry Neidenbach, Bill Schultheis and Bill Thill for their advice, inspiration and support.

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[3] D. Minoli, *Combinatorial graph complexity*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **59** (1975), no. 6, 651–661 (1976).

[4] D. Neel and M. Orrison, *The linear complexity of a graph*, Electronic Journal of Combinatorics **13** (2006), Research Paper 9, 19 pp. (electronic).

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Some common graph terminology

Acyclic: A graph is *acyclic* if it does not contain any cycles. Acyclic graphs are also known as *forests*.

Circumference: The *circumference* of a graph is the length of the longest cycle in the graph.

Complete: A graph is *complete* if every pair of nodes is connected by an edge. Complete graphs are also known as *cliques*.

Connected: Two nodes of a graph are *connected* if there is a path that links them. A *connected graph* is one in which every pair of nodes is connected.

Cycle: A *cycle* of a graph is a walk whose starting and ending nodes are the same, involving three or more different nodes. Usually, cycles are defined to be edge-distinct, meaning that the walk is not allowed traverse the same edge more than once.

Degree: The *degree* of a node is the number of edges connected to it. Every loop for a node increases its degree by 2.

Diameter: The *diameter* of a graph is the length of the longest path between any two nodes.

Directed: A *directed graph* is one in which which edges have directionality.

Distance: The *distance* between two connected nodes is the length of the shortest path joining them.

Girth: The *girth* of a graph is the length of the shortest cycle in the graph. An acyclic graph is considered to have infinite girth.

Loop: A *loop* is an edge that connects a node to itself.

Path: A *path* is a walk in which every node is visited only once.

Simple: A *simple graph* is a graph that does not contain loops, and that has at most one edge between any two nodes.

Tree: A *tree* is a connected acyclic graph.

Walk: A *walk* is a route through a graph from node to node along edges. More specifically, a walk is a sequence of nodes of a graph in which every node has an edge connecting it to its subsequent node in the sequence. Unlike a path, a node may be visited more than once in a walk.

Weighted: A *weighted graph* is a graph with numerical values assigned to each edge. These numbers are known as the weights.

Imagine Math Day: April 22, 2006

Goals:

- Allow students to take part in the creation of their own mathematics; give them a taste of what it's like to be a research mathematician
- Expose students to new mathematics, new applications of mathematics, and to the fact that mathematics is an endless discovery
- Encourage development of logical thinking, sound reasoning, good communication
- Involve teachers in an open-ended learning activity; model good pedagogy; provide teachers with helpful learning materials
- Promote the PDO group and HMC