Finally an important thing to say here: Saturday is FREE SLURPEE DAY!

10 *The Slurpee Is Free!*

Important Stuff

PROBLEM

Let's take a quick look at the function $b(n)$, which returns the sum of the squares of the reciprocals of the factors of n. For example,

$$b(10) = 1 + \frac{1}{4} + \frac{1}{25} + \frac{1}{100}$$

since the factors of 10 are 1, 2, 5, and 10.

Who says a, b, c get to have all the fun in a multi-part problem?
So, this requires your group to go through two standard deviations of thinking about this?

(h) Write out $b(4)$ as the sum of three numbers.

(u) Prove to your tablemates, beyond a reasonable doubt, that $b(4)$ is *less than* the total length of the path from yesterday's handout (the dotted path on today's handout).

(o) Write out $b(6)$ as the sum of four numbers.

(n) Again, prove that $b(6)$ is less than the total length of the dotted path.

(g) Write out $b(12)$ as the sum of six numbers. Is $b(12)$ less than the length of the dotted path?

Peg says kids should learn more about 2520, but maybe 2520 wants to maintain its anonymity.

(j) What about $b(60)$? $b(2520)$?

(e) Say, how long is that dotted path anyway? You were asked to write an expression for this yesterday.

(t) Justify this statement:

For any n, $b(n) < 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$.

So, now about that dotted path…

1. Consider the dashed path on today's handout. Which is longer, the dotted path or the dashed path? Both paths start and end in the same place.

We think the "dashed path" should instead be called "Ina's Path."

2. Find the exact length of the dashed path.
3. Prove that the length of the dotted path must be finite.
4. Prove that there is a maximum possible value for $b(n)$.

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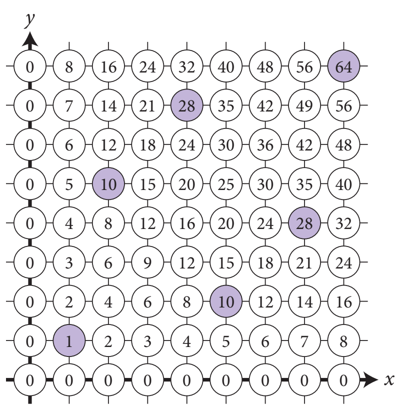
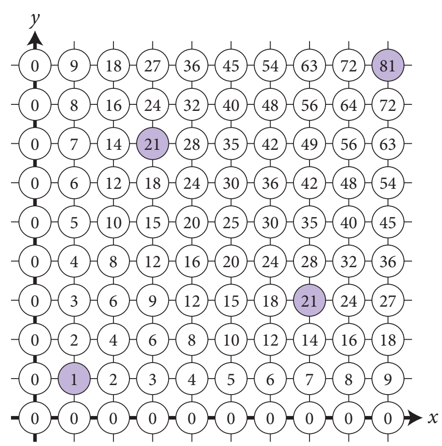
10 *Seriously, The Slurpee Is Free*

More Important Stuff

So, is this stuff really "more important" or just more? We say "more".

Last week, we defined two seemingly different functions. We defined $P(n)$, which counts how many products xy are one more than a multiple of n, when x and y are allowed to range from 0 to $n-1$. Then we defined $\phi(n)$, which counts how many numbers from 1 to n are relatively prime to n. Here are a few examples:





n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$P(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8

Golly gee willikers, the two functions give the same values for all n.

Jumpin' Jehosaphat and leapin' lizards, for the love of Pete!

5. (a) Which numbers from 1 to 9 are relatively prime to 9?
(b) In which rows and columns do the products that contribute to $P(9)$ appear?
(c) Which numbers from 1 to 10 are relatively prime to 10?
(d) In which rows and columns do the products that contribute to $P(10)$ appear?

Two numbers are *relatively prime* when they share no common factors greater than 1.

6. (a) Find an n that equals 0 in mod 3, and 1 in mod 10.
(b) Find an n that equals 0 in mod 7, and 1 in mod 10.
(c) Find an n that equals 0 in mod 9, and 1 in mod 10.

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10 *No Really, It's Free*

PROBLEM

Last Friday, we observed that the child of the $\phi(n)$ function is the identity function. For example:

$$\phi(1) + \phi(3) + \phi(5) + \phi(15) = 15 \text{ and } \phi(1) + \phi(2) + \phi(7) + \phi(14) = 14$$

Color in the pretty transparencies at your table; they correspond to the multiplication tables used to generate the P function. Stack each group of transparencies on top of each other and discuss what you see. How do the transparencies relate to this observation about the ϕ function?

Call today and we'll double your order! You'll get TWO problems in the box for the price of one. And, if you're one of the next 20 to act, we'll throw in a set of CRAZEE BONUS PROBLEMS free! You'll get zeta functions, zeta functions, zeta functions! All you pay is shipping and handling! Money back guarantee. Call now. Call now! Call now! (Seriously, we actually do have an extra problem set for those interested. It's, um, highly algebraic.)

7. (a) List all the numbers that make up $\phi(1), \phi(3), \phi(5)$, and $\phi(15)$.

(b) Multiply each set of numbers by $\frac{15}{n}$ where n is the input to each $\phi(n)$. For example, multiply all the elements of $\phi(5)$ by $\frac{15}{5} = 3$. What do you get?

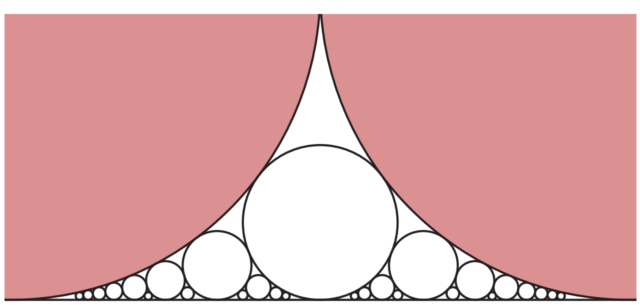
(c) Repeat for 14 by looking at $\phi(1), \phi(2), \phi(7)$, and $\phi(14)$.

(d) Repeat for an interesting number of your choosing.

Neat Stuff

8. Below is a close-up of our favorite diagram, packed with all possible circles! Between each pair of tangent circles, stuff one tangent to them and to the x-axis. Lather, rinse, and you'll end up with an infinite number of circles that are all tangent to each other *and* the x-axis. The shaded regions are the two original circles with diameter 1.

OK, the picture isn't as ridiculous as it could be. If we included an infinite number of circles, it would be. This diagram is blowed up real good!



There is one circle with diameter $\frac{1}{2^2}$ and two circles with diameter $\frac{1}{3^2}$. What other circle diameters will you find, and how many of each size will you find?

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9. Investigate a connection between the dotted path, the dashed path, and Pythagorean triples.

10. The centers of *all* the circles on the dotted path are on the same parabola. Find its equation, and sketch an accurate graph of the parabola on top of the circle diagram.

Remember it has to pass through all those centers!

11. Calculus (or a lookup table) can tell you the length of the parabola from Problem 10 from $x = 0$ to $x = 1$. Find a good upper bound for the length of the dotted path.

This isn't to say that Ina's upper bound isn't good, just that a better one can be found by using a parabola instead of using the dashed path.

12. How large can the total area of *all* the circles in Problem 8 get? Is there an upper bound, even though there are an infinite number of circles?

There's a lot of those thingies, but where are they all located?. . .

13. Explore the x-coordinates of the points of tangency of the circles in Problem 8. Specifically, if two neighboring circles are tangent at $x = \frac{a}{b}$ and $x = \frac{c}{d}$, what's the point of tangency of the stuffed-in circle?

14. Show that the left-and-right circles from yesterday's set converge on an interesting point. Your work in Problem 13 may prove very helpful.

15. Yesterday you saw the power series $p(x) = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \cdots$.

(a) Use an Nspire or any other method to calculate $(p(x))^4$. Complete this table where $f_4(n)$ is the coefficient of x^n in the polynomial power.

If this problem makes you feel ill, you might be coming down with polynomila.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f_4(n)$													
n	13	14	15	16	17	18	19	20	21	22	23	24	25
$f_4(n)$													

(b) Show that f_4 is *not* multiplicative.

(c) Find a function related to f_4 that is multiplicative.

(d) For what do you think $f_4(n)$ might be useful for?

L-functions are named after mathematician Laverne DeFazio. Often collaborating with Feeney, the first L-functions covered the numbers "5, 6, 7, 8" and many were discovered at a brewery in Milwaukee, WI. DeFazio went on to found a successful food company dealing primarily in rabbit stew.

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16. Explore the relationship between $c_1(n) = \frac{n}{\phi(n)}$ and $a_1(n) = \frac{n}{\sigma(n)}$. This table may be helpful.

n	$c_1(n)$	$a_1(n)$	$c_1(n)a_1(n)$
1	1	1	1
p	$\dfrac{p}{p-1}$	$\dfrac{p}{p+1}$	
p^2			
p^3			

Tough Stuff

17. As n gets larger and larger, what happens to the value of $c(n)a(n)$? You might prefer to ask the question about the reciprocal, $c_1(n)a_1(n)$, but it's up to you.

Don't be a ζ hata!

18. How many ways are there to write a number as the sum of two squares (including zero and negatives)? There's more than one possible formula!

19. Prove that

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^4}$$

is finite.

20. c is a number between 0 and 1. Prove that c is irrational if and only if an *infinite* number of circles from the diagram in Problem 8 intersect the vertical line $x = c$.

21. Multiply the following. What happens? Be specific! And prove it!

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\left(1 - \frac{1}{25}\right)\left(1 - \frac{1}{49}\right)\cdots$$

22. In the diagram from Problem 8, you can make a path goes from $(0,0)$ to $(1,0)$ passing through the centers of *all* the circles, in increasing order by x-coordinate. It zigzags a bit, but the length of the path through any one circle is always known. Is the total length of this path finite, or not? If it's finite, give an upper bound on the length; if it's infinite, prove it.

You can't stop this path! You can only hope to contain it.

Oh, and by the way, the slurpee? It's free. But only on Saturday.

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No Really, It's Free

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