Today's problem set is a truly moo-ving experience.

11 *Parent-Child Conference*

Important Stuff

PROBLEM

Last week, we had some practice finding the child of a function. Last Monday we considered the scintillating function $m(n) = 1$ and found its child τ and grandchild u. Today, we're going the other way on m's family tree! Suppose m is the child of a function we'll call z. Then

$$m(n) = z(\text{all divisors of } n) \text{ added together}$$

Let's work it out, starting with $z(1)$. There's only one divisor, so $m(1) = z(1)$, and $z(1) = 1$. What about $z(2)$? Well...

$$\cancel{m(2)}^{1} = \cancel{z(1)}^{1} + z(2)$$

and $z(2) = 0$. Keep going with $z(3)$ and so on. Fill in the table with the values of z, the parent of m, and then the values of moo, the parent of z.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$moo(n)$	1			0		1				1					
$z(n)$	1	0													
$m(n)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\tau(n)$	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4
$u(n)$	1	3	3	6	3	9	3	10	6	9	3	18	3	9	9

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$moo(n)$															
$z(n)$										0					
$m(n)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\tau(n)$	5	2	6	2	6	4	4	2	8	3	4	4	6	2	8
$u(n)$	15	3	18	3	18	9	9	3	30	6	9	10	18	3	27

And it came to pass that the sacred cow function moo begat z, which begat m, which begat τ, which begat u.

Careful when calculating $z(3)$, you only use the factors of 3:

$$m(3) = z(1) + z(3)$$

Similarly

$$m(4) = z(1) + z(2) + z(4)$$

Each time, fill in the ones you know... and there should only be one left!

You may feel negatively about some values you're getting for the moo function, but as long as the arithmetic works, it's all good in the Hood. East coast milk brand joke!

Parent-Child Conference

1. Go back to your notes from Week 2 and write down five things that you thought were neat, or things that you're still wondering about.

Dude! We mean business! WRITE FIVE THINGS DOWN. We're counting this as a quiz grade.

2. Find the first ten numerators in this crazee-looking product. What does this have to do with today's problem in the box?

$$\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)^2=\frac{?}{1^s}+\frac{?}{2^s}+\frac{?}{3^s}+\frac{?}{4^s}+\cdots$$

For example, when you end up seeing a term like $\frac{1}{2^s 3^s}$, write that as $\frac{1}{6^s}$. But don't try to simplify something like $\frac{2}{2^s}$ to $\frac{1}{2^{s-1}}$, just leave it so the denominators are all k^s.

3. A function is *multiplicative* if $f(ab) = f(a) \cdot f(b)$ whenever a and b don't have a common factor greater than 1.

(a) If f is multiplicative, explain why $f(5) = f(1)f(5)$ must be true.

(b) Suppose $f(5)$ is nonzero. What does the above equation say about $f(1)$?

(c) If $f(1) > 1$, explain why f *cannot* be multiplicative.

4. Define $s_2(n)$ to be the number of ways to write n as the sum of two squares, where the order and signs of numbers matters. For example, $s_2(10) = 8$ because

$$10 = 3^2 + 1^2 = (-3)^2 + 1^2 = 3^2 + (-1)^2 = (-3)^2 + (-1)^2$$

$$10 = 1^2 + 3^2 = 1^2 + (-3)^2 = (-1)^2 + 3^2 = (-1)^2 + (-3)^2$$

Fill in this table by using today's handout.

The handout should be very helpful. What shape is formed by the eight 10s on this handout? Did you know Pearl Jam's famous "Ten" album is so named because of Moo-kie Blaylock?

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$s_2(n)$	1		4								8		

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$s_2(n)$	8												

5. (a) Determine whether or not s_2 is multiplicative.

(b) Let $S_2(n) = \frac{s_2(n)}{4}$. Does S_2 appear to be multiplicative?

6. Multiply this out:

$$(1 + 2x + 2x^4 + 2x^9 + 2x^{16} + 2x^{25})^2$$

and write the terms in increasing order of exponent (so you'll write $4x^2$ before $8x^{10}$). Notice anything?

If you say "Yeah, I noticed I already did this problem last week", then keep moo-ving.

7. Define $s_4(n)$ to be the number of ways to write n as the sum of four squares, where the order and signs of numbers matters. For example, $s_4(1) = 8$ because

$$\begin{array}{ccccccccc} 1 & = & (\pm 1)^2 & + & 0^2 & + & 0^2 & + & 0^2 \\ 1 & = & 0^2 & + & (\pm 1)^2 & + & 0^2 & + & 0^2 \\ 1 & = & 0^2 & + & 0^2 & + & (\pm 1)^2 & + & 0^2 \\ 1 & = & 0^2 & + & 0^2 & + & 0^2 & + & (\pm 1)^2 \end{array}$$

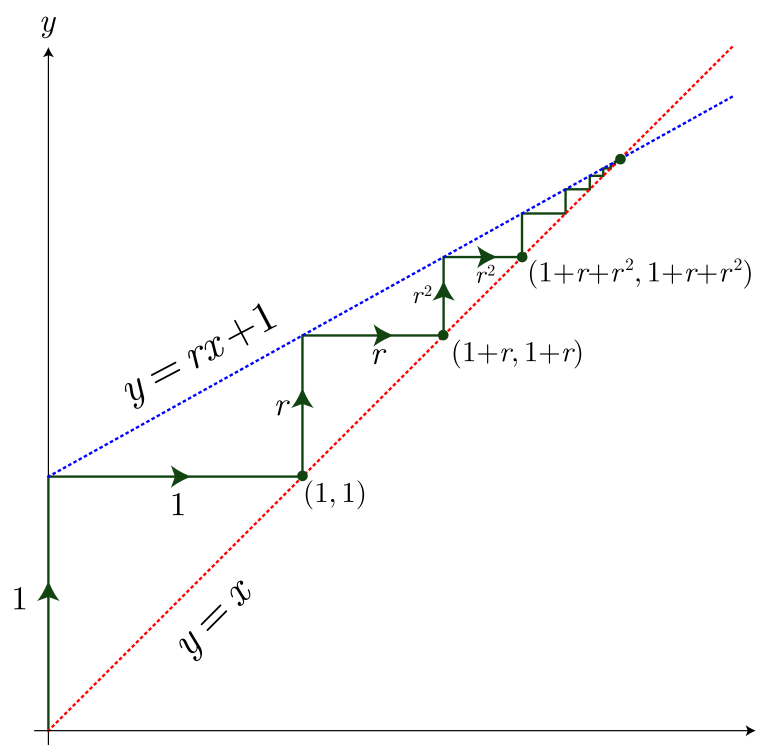
n	0	1	2	3	4	5	6	7	8	9	10	11	12
$s_4(n)$	1	8	24	32	24	48	96	64	24	104	144	96	96

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$s_4(n)$	112	192	192	24	144	312	160	144	256	288	192	96	248

(a) Determine whether or not s_4 is multiplicative.
(b) Define a function $S_4(n)$ based on $s_4(n)$ that you think is multiplicative, and test a few examples.

8. Check out this figure:

Yep, that is quite the figure. Feel free to argue about what happens when $r \geq 1$, but it's really a moo-t point.

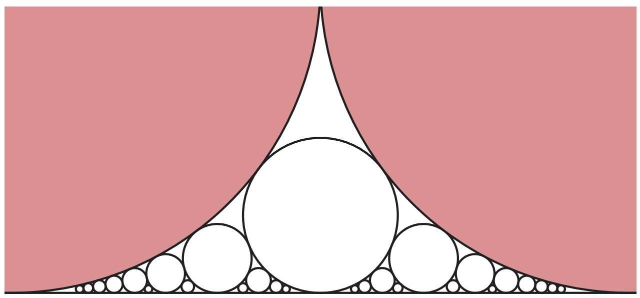


Where do the lines intersect, and what's this got to do with geometric series?

Today's problem set is a truly moo-ving experience.

Circle-y Stuff

9. Below is a close-up of our favorite diagram, packed with all possible circles! Between each pair of tangent circles, stuff one tangent to them and to the x-axis. Lather, rinse, and you'll end up with an infinite number of circles that are all tangent to each other *and* the x-axis. The shaded regions are the two original circles with diameter 1.



There is one circle with diameter $\frac{1}{2^2}$ and two circles with diameter $\frac{1}{3^2}$. What other circle diameters do you find, and how many of each size will you find?

Some of these problems are repeats, some are not. If you're tired of the circles, skip this section. But we'll be sad and moo-dy about it.

10. Investigate the x-coordinates of the centers of the circles (or, if you prefer, their points of tangency with the x-axis), especially when looking at the circles from left to right. Remember, we fixed the two large diameter-1 circles to have centers $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$.

Trivia: Who invaded Spain in the 8th Century? Answer later.

11. (a) Show algebraically that if two tangent circles have diameters $\frac{1}{a^2}$ and $\frac{1}{b^2}$, the next stuff-it-inside circle will have diameter $\frac{1}{(a+b)^2}$.

(b) Show that if a and b are relatively prime, then so are a and $(a + b)$, and so are b and $(a + b)$.

(c) Prove that if two tangent circles in the diagram above have diameters $\frac{1}{a^2}$ and $\frac{1}{b^2}$, then a and b must always be relatively prime.

Fractions help here. If a and b are relatively prime, then the fraction $\frac{a}{b}$ is in lowest terms. So if you moo-tate the fraction $\frac{a+b}{a}$...

12. Look for some Pythagorean triples in right triangles whose hypotenuses are the segments connecting the centers of mutually tangent circles. Can every primitive Pythagorean triple be found in this diagram eventually?

Neat Stuff

13. Find a rule that works for $moo(n)$ for all the n in the table earlier, especially $n = 30$. For what numbers is $moo(n) = 0$?

Tired of all these puns? The feeling's moo-tual...

14. Write out a formula for the sum of an infinite geometric series.

15. Verify each of these formulas with one or two examples.

(a)
$$1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^n} + \cdots = \frac{x}{x-1}$$

(b)
$$1 + \frac{1}{x^2} + \frac{1}{x^4} + \cdots + \frac{1}{x^{2n}} + \cdots = \frac{x^2}{x^2-1}$$

(c)
$$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$$

16. Let p be any prime. Use today's problem in the box to complete this table.

n	$moo(n)$	$z(n)$	$m(n)$	$\tau(n)$	$u(n)$
1					
p					
p^2					
p^3					
p^4					

What's a cow's favorite Bruce Willis TV series? What's a cow's favorite Cher movie? What's a cow's favorite Michael Jackson dance move? What's a cow's favorite Mets outfielder? What's a cow's favorite Cab Calloway song? What's a cow's favorite Zac Efron movie? What's a cow's favorite Stephenie Meyer book?

17. Multiply this out, again with selective cancellation (all terms should be in the form $\frac{n}{k^s}$). What does this have to do with the today's problem in the box?

$$\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)^3 = \frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \cdots$$

Notice anything?

It's not alright to shout "Nothing! Absolutely nothing!" here. Save that for Wheel of Fish, a game show where you can earn both moo-lah *and* Moo-nlight Gouramis.

18. Multiply this out:

$$\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)^0 = \frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \cdots$$

Your mathematical journey is nearing its end...

19. Determine the unique set of coefficients that make the following equation true.

$$\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)\left(\frac{?}{1^s}+\frac{?}{2^s}+\frac{?}{3^s}+\frac{?}{4^s}+\cdots\right)=1$$

Trivia Answer: the Moo-ps! Well, that's what it says on the card here.

Tough Stuff

20. Repeat Problem 4 for the number of ways to write a number in the form $n = x^2 - xy + y^2$ where x and y are integers. It'll probably help if you were here last year.

21. Prove that S_2 and S_4 are multiplicative. Try S_2 first; you may wish to use complex numbers a bit, since the norm of the Gaussian integer $a + bi$ is $a^2 + b^2$. For S_4... um, best of luck.

22. Consider a set of mutually tangent spheres on a plane. Find some relationships between the diameters of the mutually tangent spheres.

Coming soon: Spheres on a Plane! This joke is rated PG, but there is a pretty obvious R-rated version available. It starts with "Enough! I have had it..."