1 *Recurring Themes*

Hey, welcome to the class. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How did others at your table think about this question?
- **Respect everyone's views.** Remember that you have something to learn from everyone else. Remember that everyone works at a different pace.
- **Learn from others.** Give everyone the chance to discover, and look to those around you for new perspectives. Resist the urge to tell others the answers if they aren't ready to hear them yet. If you think it's a good time to teach everyone about eigenvectors, think again: the problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology: the problems should lead to appropriate uses of technology rather than requiring it. Try to avoid using technology to solve a problem "by itself". There is probably another, more interesting, way.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff, and maybe more. Check out Important Stuff first. The mathematics that is central to the course can be found and developed in Important Stuff. After all, it's Important Stuff. Everything else is just neat or tough. If you didn't get through the Important Stuff, we noticed... and that question will be seen again soon. Each problem set is based on what happened before it, in problems or discussions.

At least one problem in this course is unsolvable. Can you find them all?

Consider this your first exposure to recursion...

Every three days, go back and read these again.

Important Stuff

We're going to start with doing the same thing, over and over. The *Fibonacci sequence* is one of the most famous known in math. It starts with 0, then 1, then each new term is the sum of the two that come before it. A slightly more formal definition is

$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2) \quad \text{if } n > 1$$

For example, $F(2) = F(1) + F(0)$, then $F(3) = F(2) + F(1)$, then... Use the values of $F(0)$ and $F(1)$ to find $F(2)$, then use the values... then they tell two friends, and...

PROBLEM

(b) For the Fibonacci sequence, determine $F(0)$ through $F(9)$ and the sum of these ten numbers.

(a) Your table will be given four new pairs of starting numbers. For each pair, determine the first ten numbers (including the two givens) and their sum. Notice anything?

(h) Describe some similarities between the five sequences your table worked with.

Most of the time if we use $F(n)$ with capital F, we mean the "real" Fibonacci sequence, not these impostors.

Stuff in boxes is more important than other Important Stuff!

1. Here's a recursive definition for the sequence $0, 1, 2, 3, 4, \ldots$:

$$t(0) = 0, \qquad t(n) = t(n-1) + 1 \quad \text{if } n > 0$$

(a) For some number a, $t(a) = 23$. Find a.
(b) Calculate the sum $t(0) + t(1) + t(2) + \cdots + t(9)$.
(c) Calculate the sum $t(0) + t(1) + t(2) + \cdots + t(100)$.

2. Write a recursive definition for $a(n)$ that fits the sequence $2, 6, 10, 14, 18, \ldots$

This means $a(0)$ should be 2, $a(1)$ should be 6, and $a(73)$ should be 294. Just sayin'.

3. Write a recursive definition for $b(n)$ that fits the sequence $2, 6, 18, 54, 162, \ldots$

4. Determine the sum $a(0) + a(1) + a(2) + \cdots + a(9)$ as simply as you can, without a calculator.

Yes, that other thing is also a calculator. In general, try to do whatever you can without calculators.

5. Determine the sum $b(0) + b(1) + b(2) + \cdots + b(9)$ as simply as you can, ideally without a calculator.

6. Without a calculator, estimate the number of digits in $F(100)$, a big Fibonacci number. Yes, it's okay to get this wrong! But think it over a bit.

Avoid saying "the 100th Fibonacci number" unless it's clear what you mean. $F(100)$ is usually called the 100th Fibonacci number, but it can be confusing.

7. Find two numbers with the given sum s and product p.

(a) $s = 7, p = 10$
(b) $s = 2, p = -3$
(c) $s = -13, p = 30$
(d) $s = 10, p = 25$
(e) $s = 10, p = 23$
(f) $s = 10, p = -1$
(g) $s = 100, p = 2379$
(h) $s = 100, p = 1337$

Neat Stuff

Here are some more good questions to think about.

8. Write a recursive rule for $c(n)$ that fits the sequence $1, 2, 11, 43, 184, 767 \dots$

9. Which Fibonacci numbers are even, and which are odd? Explain why this happens.

Meanwhile, the number 7912 is weird. No, really, it is, look it up.

10. Which Fibonacci numbers are multiples of 3? Explain why this happens.

11. The *Lucas sequence* is like the Fibonacci sequence, except it starts with 2 and 1 instead of 0 and 1:

$$
\begin{aligned}
L(0) &= 2\\
L(1) &= 1\\
L(n) &= L(n-1) + L(n-2) \quad \text{if } n > 1
\end{aligned}
$$

$L(2) = 3, L(3) = 4, L(4) = 7$. There's a lot of literature on Fibonacci and Lucas. We humbly request that you not read any of it until at least the end of this week, so that you have the chance to find and prove some of the results on your own.

Find as many relationships as you can between the numbers in the Lucas sequence and the numbers in the Fibonacci sequence. Try to prove them!

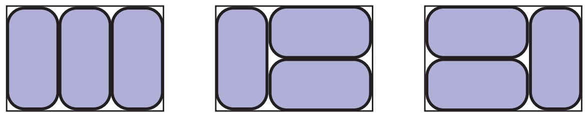
12. The *Quagmire sequence* is the sum of the Lucas and Fibonacci sequences. Alright:

$$Q(n) = L(n) + F(n)$$

Figure out what you can about the Quagmire sequence, and any new relationships you can figure out between the Lucas and Fibonacci sequences.

Giggity.

13. In terms of n, how many ways are there to tile a 2-by-n rectangle with identical 1-by-2 dominoes? Consider any rotations or reflections to be *different* tilings: there are 3 tilings for the 2-by-3 rectangle. Why look, here they are!!



14. Without a calculator, determine the units (ones) digit of $F(100)$.

15. Describe what happens with the sequence defined by

$$r(0) = 1, \qquad r(n) = 1 + \frac{1}{r(n-1)} \quad \text{if } n > 0$$

16. Some pairs of Fibonacci numbers $F(a)$ and $F(b)$ have common factors. Investigate and find something interesting about it.

Well, duh, they have the common factor 1. (We mean "legitimate" common factors.)

Tough Stuff

Here are some much more difficult problems to try.

17. Marla claims that starting with $F(7) = 13$, it's possible for $F(n)$ to be prime, but it's *not* possible for $F(n)+1$ and $F(n) - 1$ to be prime. Prove this... well, if it's true...

18. Find x if

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}}} = 15$$

19. Consider the unit circle $x^2 + y^2 = 1$. Plot n equally spaced points on the circle starting from $(1, 0)$. Now draw the $n-1$ chords from $(1, 0)$ to the others. What is the product of the lengths of all these chords?

20. Take the diagram you drew in problem 19 and stretch it vertically so that the circle becomes the ellipse $5x^2+y^2 = 5$. All the points for the chords scale too. What is the product of the lengths of all *these* chords?

Holy moly.