14 *You + Me = Us*

PROBLEM

Open this awesome Geometer's Sketchpad file:
`http://tinyurl.com/comppolygsp`

Let's investigate the function $w = f(z) = z^3 - 2z^2 - 5z + 6$.

- When we restrict the input z to a very small magnitude circle, what does the output look like?
- When we restrict the input z to a very large magnitude circle, what does the output look like?
- Explain how you know that there *must* be a magnitude circle that contains a point z with $f(z) = 0$.

I know my calculus. . . it says you plus me equals ganas. . .

Come on, let's!

Embiggen your circle! Biggen!

Amazingly Fantastic Stuff.

1. David says he can use the method of the box problem to show that, for *any* polynomial, there is at least one z with $f(z) = 0$. He's just nuts, right?

Your Stuff.

2. A tetrahedron has three of its faces meeting together at right angles. Find a relationship between the areas of its four faces.

It's like a corner of a cube sliced out. Take that, Pythagoras!

3. Judi, Dan, Cuong and Katherine are walking to 7-11 located at point $(7, 11)$ at 7:11 p.m. (on July 11th for free Slurpees, of course). Dan bet Katherine she couldn't find the area of the triangle created by our trip from Prospector $(86, 2)$ to 7-11 at $(7, 11)$ and back to her room $(7, -7)$. Did she collect? What is the area?

Free Slurpees may be available on other days, but only in rare circumstances.

4. Faynna, Joey, and Danielle are standing at the vertices of triangle FJD. At what point A should Anna stand to minimize $FA+JA+DA$? Let L be this minimum distance. Define Anna's badness, $B(A)$, to be $FA+JA+DA$. What does the locus of points of a fixed badness look like? What if Anna stands outside of triangle FJD? What is the locus of points with a fixed $B(A)^2$?

Clearly, Anna should stand at the nearest Chili's for FAJADAs. And, no, Anna is not actually bad, except possibly in the Michael Jackson or Domenico Modugno sense.

5. How can you prove that the 120-degree point has the shortest total distance to the three vertices in a triangle? Does your proof work for *any* triangle?

6. Find two non-congruent triangles with integer side lengths that share the same area, A, and the same perimeter, P.

7. Start with $z = 1 + i$ and graph (sketch) several positive integer powers of it. Then do the same thing, but starting with $w = \frac{1}{2} + \frac{1}{2}i$. What is the value of the infinite sum $1 + w + w^2 + w^3 + \cdots$?

8. Find all positive integer solutions to the equation below, where $A \leq B \leq C$.

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = \frac{2}{3}$$

If you find a connection between this problem and others from these sessions, let us know, because we looked!

9. A more general version of the unit fraction problem is

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = \frac{k}{2}$$

where n, k, and all x_i are positive integers. Find some conditions on n and k for which this problem has a solution.

10. Go back to problem 5 from Day 13, but now suppose that Mike, Mariam, Matthew and Monty are arranged in a rectangle rather than a square. How does the solution change? If they are arranged in a general quadrilateral, will the solution be unique? What if there are five points in an arbitrary arrangement?

The four people are friends so they will be *MMMM*. When they have a satisfying meal, they go *MMMM*. Their favorite Crash Test Dummies song is *MMMM MMMM MMMM*. If they don't know the words at karaoke, they *MMMM*. And when Heron joins them to make a pentagon, they go *MMHMM*.

11. Three regular polygons can meet at a point with no overlap and no empty space in several ways. But which ones tessellate the plane?

12. For each regular -gon, find the side length that will make its area numerically equal to its perimeter.

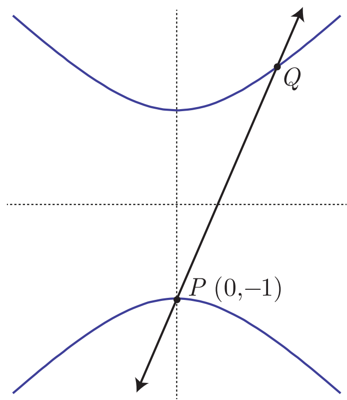
(a) 3-gon
(b) 4-gon
(c) 6-gon
(d) 8-gon
(e) 12-gon
(f) n-gon

13. Given a specific surface area and volume, how can we find all the boxes with that surface area and volume? What if we are restricted to boxes with integer side lengths?

14. Can every Heronian triangle be deconstructed into two Pythagorean triples?

15. On Day 4 we intersected the unit circle $y^2 + x^2 = 1$ with a line of slope m passing through $(0, -1)$. The other intersection with the unit circle could be found in terms of m, and magical things happened: we could find as many rational-coordinate points on the unit circle as we wanted! Repeat this process with the unit *hyperbola* $y^2 - x^2 = 1$ with a line of slope m passing through $(0, -1)$. Find the other intersection in terms of m and enjoy the magic.

It's *magic*! If you squint hard enough, the hyperbola transforms into a unicorn.



16. Compute each value. Find all possible answers!

(a) i^2
(b) i^3
(c) $\sqrt{i}$
(d) $\sqrt[6]{i}$
(e) $(3 + 4i)^2$
(f) $\sqrt{3 + 4i}$
(g) $\sqrt{3 - 4i}$

DOES NOT COMPUTE. Meep meep meep.

17. Find another complex number whose direction is 60 degrees counterclockwise from these given complex numbers, as seen from the origin.

(a) $1 + i$
(b) $3 - 2i$
(c) 5
(d) $a + bi$

What are the roots of that thing with two numbers that add to whatever that thing is? Eh, you know what we mean.

18. Find a third point in the coordinate plane that will complete an equilateral triangle with these given two points.

(a) $(3, 4)$ and $(4, 5)$
(b) $(2, 1)$ and $(5, -1)$
(c) $(3x, 2)$ and $(3x + 5, 2)$
(d) (x, y) and (w, z)

Psst: the last problem might help you! Or it might not! OH ho ho ho.

19. The "taxicab distance" (a.k.a. "Manhattan distance" or "rectilinear distance") between any two points in a plane is equal to the sum of the absolute differences of their coordinates. If you think of the lattice points on a graph as being the intersection of streets and avenues, the taxicab distance between two points represents how far a taxi must go to get from one intersection to another *travelling only on the streets and avenues* of the plane. The taxicab distance between $(3, 5)$ and $(7, 3)$ is calculated as $|7 - 3| + |5 - 3| = 4 + 2 = 6$. Note that there are several paths between the two points that have this minimum path length.

In Park City, the taxicab distance is usually greater than the walking distance, and may even be slower at night.

(a) Jason is a fireman in brand new Gridville, which only has three houses at $(1, 4)$, $(6, 1)$, and $(9, 5)$. Where should his firehouse be built to minimize the average taxicab distance to the three houses? This Geometer's Sketchpad file may be useful:

`http://www.tinyurl.com/taxicabgsp`

Jason wanted to grow up to be a bow-tie-wearing lounge singer, or perhaps a night sky photographer, but both of these options fell through. So he's stuck as a fireman.

(b) Two more houses are built in Gridville at locations $(2, 7)$ and $(3, 6)$. Fortunately, Jason's firehouse is easily moved. Where should he move it?

(c) Stop and reflect on where these "best" points are and how they relate to the locations of the houses. Notice anything?

(d) A sixth house is built at $(6, 6)$. Find a new firehouse point for these six houses. And more! How many are there? Did you find six of them? Wait... why not?!

We aren't sure if Jason is telling the truth here, or if his pants are en Fuego.

(e) Farmville crops up nearby with two neighborhoods of houses. The houses are located at $(2, 8)$, $(2, 9)$, $(3, 7)$ and $(3, 10)$, as well as at $(5, 1)$, $(7, 4)$, $(8, 2)$, and $(10, 5)$. Find all the firehouse spots for this set of houses. Good luck and good night.

Or, in Jason's case, good morning! Be sure to wake him up!

20. Go back through the previous problem and find firehouse spots that have the lowest possible *maximum* taxicab distance to any house. How are these points related to the median?

21. There may or may not exist in the real world a *unicorn tetrahedron*. These are special tetrahedra defined by having integer edge lengths, integer areas of faces, and integer volumes. If they exist, and you construct one out of paper, you will see a magic unicorn—sparkling with glitter. Please, please, for all that is good in this world, show that in our world, a *unicorn tetrahedron* exists. Or be cruel and dash Bill's hopes and show that they don't.

Man, what were we smoking when we wrote this problem? Oh, right, *one of you* wrote it.

Which Bill are we talking about? There is Bill and then there is B̈ill who spells his name with an umlaut.

22. Is it possible for a clock to have 120 degrees between all three hands if it is

(a) a continuously-moving second hand?

(b) a clock with discrete second "ticks" (with minute and hour hands moving incrementally)?

23. Is it possible to spoken-word sing both "Goodnight Irene" and Jimmy Eat World's "The Middle" in the span of two minutes?

Gertrude would know the answer to this.

Our Other Stuff.

24. Yesterday's arm-waving guy with the accent and the soap said that for a regular octagon, the shortest path is made by using the boundary instead of creating interior "Steiner points". Do you see any particular reason why this would be?

25. Go back to Brian's data from problem 14 on Day 11. Suppose $h = mt + b$ is an attempt at a best fit line and its badness is measured using the sum of absolute errors. In the m-b plane, what is the locus of all points (m, b) corresponding the same badness?

26. Jemal asks you to close your eyes and imagine a four-dimensional hyper-box with dimensions A, B, C, and D. (He's really good at this. Ask him how.) Find all possible dimensions for this hyper-box so that $A \le B \le C \le D$ are integers, and its four-dimensional hypersupervolumey-thing has the same numerical value as its total "face-volumes." Oof, my head hurts.

Jemal also asks you to imagine his driver's license expires in 2012, but we know that's just fiction.

27. When's the next time we'll see you again? Thanks for playing and thanks for teaching.