Algebra PoW Packet

Filling Glasses

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Welcome!

This packet contains a copy of the problem, the "answer check," our solutions, teaching suggestions, a problem-specific scoring rubric, and some samples of the student work we received when *Filling Glasses* was used in January 2005.

We invite you to visit the PoW discussion groups to explore these topics with colleagues. From the Teacher Office use the link to "PoW Members" or use this URL to go to *algpow-teachers* directly: http://mathforum.org/kb/forum.jspa?forumID=528 [Log in using your PoW username/password.]

The Problem

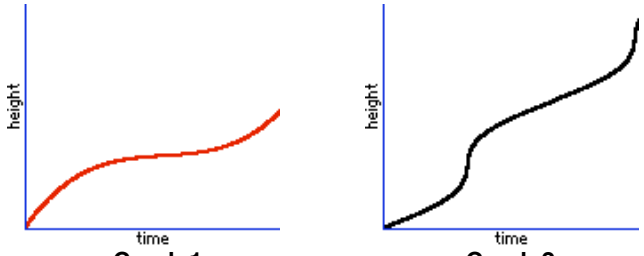
We are revisiting *Filling Glasses*, Problem 2848 from the Library. It's a nice problem for early in the school year as it gets kids thinking about graphs and the relationship between quantities but does not require any formal algebraic knowledge to solve. There are really three important mathematical elements to the problem: the math conditions that drive the event (the constant water rate and the shape of the glass), the math extracted from the event (the height of the water over time), and the math of the graphs (what does the shape of the curve represent). The key to solving the problem is to be able to connect those three elements.

The problem also gives rise to a number of possibilities for interesting exploration and discussion, including graph scales, graphing in general, rates of change, instantaneous slopes, functions, and modeling. Suggestions on introducing the problem to your students appear in the Teaching Strategies portion of this packet. Here's the problem:

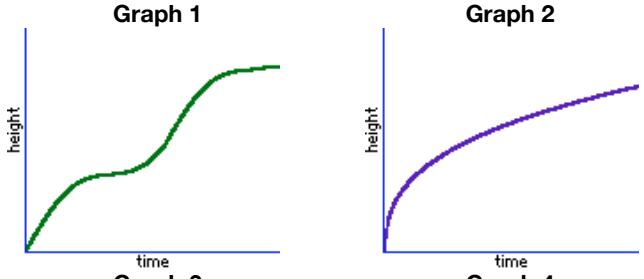
If we were to fill a glass with water at a constant rate (for example, 1 cubic inch per minute), we could graph the height of the water in the glass as time goes by.

Suppose we fill the three glasses below in such a manner. Match up each glass with the graph that best describes the height of the water in the glass over time.

Glass A **Glass B** **Glass C**



Graph 1 **Graph 2**



Graph 3 **Graph 4**

This problem may seem different from the types of problems that you're used to, but don't assume that that means you can't do it. Think about how the shape of the glass changes the rate at which the water rises. If you're stuck, no problem. Tell us what you are thinking and make sure to leave a comment about the help you want.

Extra: Describe the glass that would correspond to the leftover graph.

Answer Check

After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually **get** the answer, and in this case only tells how many solutions there are to each equation), we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation:

Glass A is best described by graph 1.
Glass B is best described by graph 4.
Glass C is best described by graph 3.

If your answer **doesn't** match our answer,

* imagine actually doing this - when would the water level in the glass rise fastest? When would it rise more slowly? How would that show up in a graph?
* remember that the graphs are showing height versus time.

Still stuck? Tell us as much as you can in your comment about where you think you're stuck, and we'll try to give you a hand.

If your answer **does** match ours,

* have you carefully explained how you decided on those matchups?
* are you confident that you could solve another problem like this successfully?
* Did you try the extra?

Our Solution

There is less formal math than usual in this problem. In fact, it's possible to answer it completely without using any numbers at all! But that doesn't mean that there isn't math going on behind the scenes. The key idea, and the one thing they need to explain, is how the graphs relate to the shapes of the glasses. It might look something like this:

> As you pour liquid into a container at a constant rate, the speed with which the height of the liquid in the container rises varies depending on the shape of the container. Aiming a hose spouting water into a 4 foot kiddie wading pool is going to result in much less rapid height increase than aiming that hose into a drinking glass, for instance!

The key dimension of the container is its diameter. Of course, since not all containers are round and have diameters, we might generalize that a bit and say that it depends on the cross-sectional area of the container. Let's imagine that in a given second, the filling hose emits a certain volume of water. So we're interested in what that particular volume will look like in the container at that moment in time.

We generally think of volume of a three-dimensional object as cross-sectional area (or area of the base) times the height. For a fixed volume, as the area gets smaller, the height must grow larger. We could even say that the area and height vary inversely - as one increases, the other decreases, maintaining a constant product for the volume.

Applying this to the glasses, we can see that the wider the glass is (the greater the diameter and cross-sectional area) the less height will be needed to hold the given volume per unit of time. Thus, the wider the glass is, the slower the height of the water rises. The narrower the glass is, the faster the height of the water rises.

Turning our attention to the graphs, note that they show time versus water height. As time moves along the bottom axis, the height of the water is indicated on the left axis. In a case where the height is increasing slowly, and therefore not changing much over time, the graph will be "flatter". When the height is increasing quickly, it will be "steeper". With these ideas in mind, let's look at the glasses and graphs.

Glass A grows wider at the bottom, so the height will rise less rapidly and the graph should curve towards more level. In the middle section it continues to get wider but much more slowly, so the graph should be rising slightly. Near the end, the glass gets slightly narrower, so the graph would rise more quickly and turn upwards. Graph 1 most closely resembles this pattern.

Glass B is basically a circle of steadily increasing diameter due to its cone shape. The graph would rise less and less quickly, falling more and more toward level, and it would do it steadily without any sudden twists or turns. Graph 4 most closely resembles this pattern.

Glass C starts by getting wider, meaning the graph should be turning towards level. But then it gets narrower again, so the graph would turn steeper. Then it grows wider again, so the graph would once again turn down towards level. Graph 3 most closely resembles this pattern.

Extra: Graph 2 starts out almost linear, which would mean the water is rising at a constant rate and the container would be cylindrical. Then the graph shoots upward, meaning a rapid narrowing of the container. Then it comes back down to the original steepness and maintains that for a while, so back to the cylinder with same diameter as earlier. Finally, at the end it shoots up again as the container narrows once more. I envision a cylindrical container with an indentation around the middle, perhaps for holding it, and then a narrowing at the top, perhaps for a nozzle.

OK, I recognize that few if any kids are going to write something that sounds like this! But what I'd really like them to talk about is the relationship between volume, area, and height, and how the graphs reflect the narrowing and widening of the container over time. The more elegant the comments the better, but those are the topics that should be touched on.

Teaching Suggestions

This is a great problem for physically acting it out. Provide or have kids bring in glasses with unusual and various shapes. Turn a faucet on so that it flows at a fairly steady rate, and have the kids hold the different glasses under the stream to watch what happens. Have them focus on how rapidly the water level in the glass is changing in relation to the shape of the glass. They may need to experiment to find a flow rate that most dramatically illustrates the different rising rates in the glasses. Many kids will have a good intuition about a narrower glass filling more quickly and will be able to confirm that intuition with an activity like this.

Have kids sketch graphs of water height versus time that represent the results they saw in their glasses. This will give an opportunity for them to talk about how to indicate a faster or slower rate on the graph, and perhaps some will even talk about slope, though they may not be aware that slope can be measured at any point on a graph if their experience with slope has been limited to straight lines.

Once kids have found or confirmed the relationship between shape and rising rate and thought about how the rate is represented on a graph, they should be able to solve the problem successfully. At that time, it would be great to encourage discussion about the math behind the relationship between shape

and rate. Why does a narrower glass lead to a faster rising rate? Ask questions that encourage them to think about volume, but resist steering them directly to it. Kids could come up with a good explanation without using the word "volume", such as saying that in a wider glass there is more room for the water to spread out, so it doesn't rise as quickly. The goal is to have them think about what it is that relates shape and rising rate, what it is that makes their intuition or observation correct.

Scoring Rubric

The **problem-specific rubric** is something we write for every problem for use by those who are assessing student work. It spells out what we expect from students in three areas of problem solving and three areas of communication. The goal is to assess a student response within each category as it relates to the specific criteria for that category. This approach allows you to retrieve more targeted information on the students' areas of strength and weakness.

In the most general sense, *Interpretation* usually includes understanding the given information, including any diagram, attempting to answer all parts of all of the questions asked, and exhibiting understanding of any fundamental math concepts necessary to solve the problem. *Strategy* is usually then applying all of that knowledge in a systematic and mathematically sound way that doesn't rely on any lucky guesses. *Accuracy* simply refers to executing your strategy correctly. Note that even an incorrect strategy can be done accurately. *Completeness* is showing and explaining the thinking and work you did to reach your answer, and *Clarity* is presenting that explanation in a way that is easy for someone else to read and understand. *Reflection* includes such things as thinking about the reasonableness of your answer, checking it, tying the problem to past problems, and summarizing the key ideas in your solution.

A **generic student-friendly rubric** can be downloaded from the *Scoring Guide* link on any problem page. We encourage you to share it with your students to help them understand our criteria for good problem solving and communication.

The problem-specific rubric is shown on the next page, followed by sample student solutions from when *Filling Glasses* appeared in January of 2005.

Happy Problem Solving!

We hope these packets are useful in helping you make the most of the AlgPoWs. Please let me know if you have ideas for making them more useful.

– Ríz

riz@mathforum.org

Algebra Problem of the Week Scoring Rubric for *Filling Glasses*

For each category, choose the level that *best describes* the student's work.

	Novice	Apprentice	Practitioner	Expert
Problem Solving				
Interpretation	shows understanding of only one or two of the concepts involved - see the Practitioner column	shows understanding of most but not all of the concepts in the Practitioner column (for example, goes through a process of matching glasses and graphs, but misinterprets the relationship between time and height shown in the graphs) does not complete all parts of the problem	understands that water is pouring into the glasses at a constant rate understands that the shape of each glass generates a unique graph of water height versus time as it fills understands how to interpret the given graphs in terms of time versus height of water in glass, whether seeing "height" as depth of water in bowl or distance between water line and bottom of the stem	describes or draws an appropriate glass shape to match the unused graph for the Extra
Strategy	has few ideas that will lead them toward a successful solution	picks an incorrect strategy, based on their interpretation of the problem or relies on luck to get the right answer (note that if they don't understand the problem correctly, but choose an appropriate strategy for their misinterpretation, they should be Practitioner, not Apprentice)	picks a sound strategy based on interpretation of the problem—success achieved through skill, not luck connects wide/narrow glass shape to flat/steep graph shape interprets steep/flat shape of graphs correctly understands relationship between constant rate, volume of glass, diameter of glass, and rising rate	mentions that graph B is the top half of a right-facing parabola and ties that into the quadratic nature of the area of a circle mentions that cross-sectional area and height vary inversely for a fixed volume and how that relates to this problem
Accuracy	work contains many errors	work is mostly accurate, with a few errors	work is accurate, contains no errors, and uses correct vocabulary	generally not possible - can't be more accurate than Practitioner
Communication				
Completeness	has written very little that tells or shows how they found their answer	explains only one or two of their choices completely doesn't include enough information for another student to follow their explanation	explains most of the thinking they used to solve the problem, which might include… how they interpreted the shapes of the graphs how they interpreted the shapes of the glasses how they connected the glasses and the graphs	adds in useful extensions and further explanation of some of the ideas explains why a narrower glass fills more quickly than a wider glass
Clarity	explanation is very difficult to read and follow	explanation isn't entirely unclear, but would be hard for another student to follow explanation is long and is written entirely in one paragraph explanation contains many spelling and typing errors	explains all of the steps in such a way that another student would understand makes an effort to check their formatting, spelling, and typing (a few errors are fine as long as they don't make it hard to read)	formats things exceptionally clearly answer is very readable and appealing adds a diagram or picture for the Extra to make the glass shape clear
Reflection	The items in the columns to the right are considered reflective. They could be in the solution or the comment left after viewing the Math Forum's answer.	checked answer in some way (in addition to viewing the answer provided by the Math Forum) reflected on the reasonableness of their answer	connected the problem to prior problems or experiences explained where they are stuck summarized the process they used	commented on and explained the ease or difficulty of the problem revised and improved their work
	did nothing reflective	did one reflective thing	did two reflective things	did three or more reflective things *or* did an exceptional job with two of them

Student Solutions
With focus on Completeness

Completeness means fully explaining the thinking and work you did to solve the problem. In this problem, it might mean talking about how you interpreted the graphs with regard to flat and steep and how the shapes of the glasses lead to different rates of rising for the water level. Note that you do not need to have chosen a correct strategy or solved the problem correctly to be a Practitioner in Completeness - you can explain your incorrect work very completely and in a way that someone else can understand exactly what you did.

The sample student solutions included in this packet represent a broad range of both writing and problem solving skills. They also show a range of understandings, and we've tried to address each student's individual misunderstanding or weakness with comments that suggest what might be a good next step for that student. For those who have scored Novice or Apprentice in Interpretation, we've focused on their understanding of the problem. For those who are a Practitioner in Interpretation, we've focused on elements of their communication that could be improved.

Taylor
age 13

Completeness
Novice

The answer to the problem is that graph 1 belongs to glass A, graph 4 belongs to glass B, and graph 3 belongs to glass C.

We found this out by looking at the graphs and the glasses and seeing the pattern between the two. We did this by comparing the shape of the glass and the line in the graph. That is how we figured it out.

Taylor hasn't said much of anything that helps us understand how she came up with her answer. I'd ask her to tell me more about the pattern that she saw.

Max
age 13

Completeness
Novice

The solution is that glass A goes with graph 1, glass B goes with graph 4, and glass C goes with graph 2.

Glass A goes with graph 1 because it is narrow, then wide, then narrow. Glass B goes with graph 4 because it is narrow then wide. Glass C goes with graph 2 because it is wide, then narrow, then wide.

Max seems to have done some good noticing about the glass shapes, but has not explained how he linked those shapes with his graph choices. I'd ask him to tell me more about how the narrow and wide shapes would affect the rate at which the water would rise.

Andre
age 13

Completeness
Apprentice

For glass A we got graph 4. For glass B we got 3. For glass c we got graph 2.

To find our answer we found the circufrance and figured the cubic inches and capacity.

Andre is hinting at some interesting math, but has not said enough about it for us to understand how he found the things he mentions or how he used them to match glasses and graphs. His answers are incorrect, so I'm pretty sure he's confused. I'd ask him to first tell me what he notices about the shapes of the glasses and the shapes of the graph without worrying about trying to match them yet.

Zach
age 14

Completeness
Apprentice

Graph 1 is A, graph 3 is class C, Graph 2 is blank and Graph 4 is B

For graph 4 has a constant slope and graph B goes at a constant rate so we matched the glasses and the graph. Graph 1 goes fast slow fast and glass a goes skinny wide skinny so they matched and finnaly graph 3. It goes Fast slow fast slow and glass C goes Wide Skinny wide skinny so they matched giving us the answers above.

Zach has done some nice work and it's clear that he's thought about the shapes of the glasses and the graphs. I'd ask him to tell me more about what he means by fast and slow on the graph and how he used skinny, wide, fast, and slow to make his choices.

Andreas
age 15

Completeness
Apprentice

Glass A is Graph 1, Glass B is Graph 4, and Glass is Graph 3

When the time line goes vertically, it means the shape of the glass is narrow. It will fll up faster. When the time line goes horizontally, it means the shape is wider. It will fill up slower.

Andreas has made some nice general observations about the graph shape, but has not said anything about the shapes of the glasses. I'd ask him if he can tell me something about the glasses along the lines of what he's done with the graphs. Then I'd encourage him to explain how he used those observations to make his choices.

Barbara
age 14

Completeness
Apprentice

If it is supposed that each of the three glasses pictured above are filled with water at a constant rate, then Glass A would match up with Graph 1, Glass B would match up with Graph 4, and Glass C would match up with Graph 3.

To solve this problem, the first step I took was understanding the graphs. I recognized that the height of the water in the glasses was on the Y axis and time elapsed was on the X axis.

I then looked at each graph individually and tried picturing the glass that the graph was charting in my mind. Then I returned to the picture of the glasses and tried matching the glass in the picture with the graph.

Once I thought I had the correct answers, I double-checked my solutions by reversing the way I originally looked at the problem. Instead of scrutinizing each graph, I observed each glass and tried picturing the corresponding graph.

Extra: The remaining graph (Graph #2) should illustrate a glass with a wide base because at first the line does not ascend quickly in relation to time. Then the glass should narrow because the line begins to ascend more steeply in relation to the time. After this, the line grows more horizontally, representing that the glass must be getting wider. The graph finishes with a very steep ascension of the line meaning at the top of the glass, it must be narrow.

Barbara's explanation of her thinking on the Extra is excellent. Her main explanation, though, does not offer any of that detail. I'd ask her to work on explaining her decisions on the main problem with the same detail that she used on the Extra.

Dylan
age 13

Completeness
Practitioner

Glass A matches up with graph 1. glass B matches up with graph 4 and finally, glass C matches up with graph 2.

We looked at which graph was tallest and we mathced it up with with the tallest glass. We did the same by looking at height and matching them up with the glasses and graphs.

Although Dylan has misunderstood the problem, his explanation of how he solved it is fairly complete. He simply matched the heights of the glasses with the final heights of the graphs, ignoring the shapes and curves entirely. I'd ask him if he can be sure that the scale on all the graphs is the same and if it makes sense to compare the heights without a scale.

Trevor
age 15

Completeness
Practitioner

Glass A fits with Graph 1, Glass B fits with Graph 4, and Glass C fits with Graph 3.

In the graphs, the vertical axis is the height, and the horizontal axis is the time it takes. So, the steeper the slope of the line, the faster the glass fills up, because it reaches a greater height in the less amount of time.

By looking at the glasses, we know that the less wide they are, the faster they will fill, and the more wide, the slower they will fill. So, because a steeper slope represents it filling faster, the wider parts of each glass will have a less steep slope than the other parts.

Graph 1 starts out steep, levels off, and then becomes steeper again. So, that must mean it starts out less wide, becomes wider, and then less wide. This description fits with Glass A.

Graph 2 starts out at a slope, then becomes much steeper, then less steep, then more steep again. This means that it starts off wider, then less wide, then more wide, then less wide. This description doesn't fit with any of the glasses it gives us.

Graph 3 starts out at a steep slope, levels off, becomes steep again, and then levels off again. This must mean it starts out less wide, then becomes more wide, then less wide, and then more wide again. This description fits with Glass C.

Graph 4 starts off very steep, and then slowly becomes less steep. This must mean the glass shape starts out very thin, then slowly becomes more wide as it gets higher. This description fits with Glass B.

Extra: The leftover graph is Graph 2, and the glass that fits it would have a wide bottom, then becomes less wide, then more wide, and then less wide as it gets higher.

Trevor has done an excellent job of explaining the thinking he used to come up with his solution. The reader is left in no doubt as to where his answers came from. Trevor clearly understands the problem thoroughly, so I'd ask him to think and talk about the math that's going on behind the scenes. Why does a narrow glass fill more quickly? Note that Trevor would also receive an Expert score in Clarity - his solution is formatted nicely, very easy to read, and easy to understand.

Emma
age 13

Completeness
Practitioner

The matching goes as follows: Glass A and Graph 1. Glass B and Graph 4. Glass C and Graph 3.

What a strange question on math forum! But a fun one, too! Ok, the way that I did this problem was to look at the charts to see which shapes were possible. I emidiately rulled out graph 2, because none of the glasses started in a pointed bottom. All of them have rounded bottoms. Then, It was just a matter of matching the 3 graphs and 3 glasses.

I matched graph 1 to glass A because by looking at the shape of the graph line, you can see where certain curves of the glass are. Looking near the base of the red line, you can see that the begining/bottom of the glass starts on a rounded and narrow way, and bulge out, then curve back narrower, but still wider than the begining. We can also see that this glass is the shortest, but it is also wide. Glass A is over all the smallest container of them all. It will hold the least water. And it matches the profile for the graph perfectly.

Glass B is matched to graph 4 because Glass B is the tallest and the straightest of them all. It has a rounded bottom, and straight sides. Since it has no curves, we can rule out graph 3, so the only remaining graph is graph 4. We can tell that the glass for graph 4 is tall, it fills up evenly and the water fills more space as it goes, suggesting a glass that gets wider as you go. Therefore, Glass B goes to Graph 4.

Now, our last Glass is Glass C. The only remaining graph is graph 3, but I still need to check this answer. Since graph 3 portrays a flattened bottom, and a deep curve, and gets wider as it goes, we can match this to glass C.

As for the extra, the graph could correspond to a glass similar to glass B, but narrower and with a pointed bottom. Yay! Im done !!!

Like Dylan above, Emma is a great example of how a student can get low marks in Interpretation but still score well in Strategy and Completeness. She has not really understood the problem, but has done a good job of explaining how she solved it based on that misunderstanding. Her comment about graph 2 not matching because no glass has a pointed bottom gives good insight into her thinking. She has apparently tried to match the edge profile of the glass with the shape of the graph, albeit with the graph representing the glass lying on its side. That's how she views graph 2 as representing a glass with a pointed bottom and the other three as representing glasses with rounded bottoms. I think Emma and Dylan would both be good candidates for the sort of exploration mentioned in the Teaching Strategies section of this document, to help them get a basic handle on the problem.

Eric
age 14

Completeness
Expert

Graph 1 is for Glass A, Graph 4 is for Glass B, and Graph 3 is for Glass C.

To solve this POW, I used logic and anaylzation of the glasses and graphs. I assumed that the the graphs would not be perfectly accurate, but accuracy wasn't necessary, because the change in the slopes in the graphs were the main factor in finding which graph matched which glass. When the width of the glass got smaller, the slope of the graph would become steeper, and when the width was wider, the slope was flatter. This is because when a glass is wider, it takes more water to fill, and therefore, takes longer to get higher. This was the method I used to determine my answers.

Graph 1 represents Glass A because the glass starts out thin then gets wider and then thinner near the top. The graph has a slope which starts out steep then gets flatter and then steeper.

Graph 4 represents Glass B because the glass constantly becomes wider. The graph has a slope which becomes constantly flatter.

Graph 3 represents Glass C because the glass starts out thin, becomes wider, then thinner, and then wider. The graph follows accordingly by starting out with a steep slope, then going into a flatter slope, then a steeper slope, and finally a flatter slope.

This POW problem was similar to a previous problem about voting. This POW was very easy and just took some thinking and logic.

In addition to completely explaining his thinking about the glass and graph shapes and the correlations between them, Eric has also included his thinking about why a wider glass would lead to a slower rise in water height. We're left with no question as to how Eric came up with his solution to the problem and why he thinks it's right.