



The Math Forum: Problems of the Week

Problem Solving and Communication

Activity Series

Round 14: Work Backwards

Working backwards is a particularly useful problem-solving strategy when you can clearly define the goal or end state of the problem, and you know a sequence of operations that were used in the problem. Reversing the operations and working backwards from the goal helps problem-solvers to describe the initial conditions or the most efficient path to the goal state. Working backwards is often applied to logic problems, like the famous one about crossing the river with a cabbage, a goat, and a wolf (<http://mathforum.org/library/drmath/view/59217.html>), in which you know the goal state (everything on the opposite side of the river), and you know what the legal moves are (rowing one animal across the river without leaving the cabbage with the goat or the wolf with the goat). You can work backwards from the goal, asking yourself, what must have been the last animal rowed across the river? What must have happened just before that?

Another sort of problem that working backwards can be applied to is a problem involving operations on a quantity, in which you know the final outcome after all the operations have been applied, and you need to find out the initial quantity. In these sorts of problems, it's useful to play the situation backwards, performing the inverse of each operation on the known, ending quantity until you are left with the initial quantity. This process of inverting operations is very similar to the process of solving an algebraic equation by "undoing" what's been done to the variable.

The activities below help students to clearly define the goal states and operations, to imagine reversing the operations, and to check their solutions in a variety of ways.

The activities are written so that you can use them with problems of your choosing.

Problem-Solving Goals

Working backwards can help problem-solvers:

- Discover the mathematical relationships, similar to what you do with the *Guess and Check* strategy without having to guess.
- Find unknown initial conditions when goal states are known.
- Choose the most efficient among multiple paths from initial state to end state.

Communication Goals

Problem-solvers who work backwards might use the writing process to:

- Organize and keep track of complicated sequences.
- Describe the problem context in terms of the quantities and relationships that can be "unwound" or worked backwards.
- Facilitate the transition from verbal descriptions to mathematical representations.
- Make one's thinking available for comparison and testing.

Activities

I. Describing the Steps (First to Last)

Format: students working individually or in pairs.

In order to work backwards, it's important to identify a number or final result at the end of a process that you will "unwind" or "undo" as you work backwards. What are some of the answers or final results that the problem tells you at the end of the situation? What happened, and in what order?

Sample Activity

Work individually or in pairs on the following activities for just a few minutes. You might answer the questions using pictures, a table, arrows, symbols, or words. Try to be as succinct and mathematical as possible. You want to use as few words as possible so that you can focus on the quantities and what was done to them.

- 1) What are the steps in the problem? List them in order, from start to finish. Be sure to represent the operations or actions that are done to move from step to step.
- 2) What numbers or final results do you know that might be a good place to start working backwards?

Key Outcomes

- Identify the “goal state” of the problem and any final numbers or results that might be used to work backwards.
- Identify the operations that immediately preceded each state of the problem.
- Writing the steps and operations out clearly so you can undo or unwind the problem.

II. *Playing the Movie Backwards*

Format: In pairs.

Many problem-solvers describe working backwards as playing a movie of the problem backwards in their head. In reverse, giving becomes taking, doubling becomes halving, etc. See what your problem looks like on rewind.

Sample Activity:

Each partner tries to visualize the steps they've written happening in reverse, as though they were watching a movie of the problem backwards.

Take turns describing the action in each step to your partner backwards. Use inverse operations and quantities whenever possible.

For example, if one step (going forward) was “Meg gives 3 apples to Sally,” then in reverse you might visualize 3 apples moving back from Sally to Meg. The inverse of “give” might be “take” and going backwards it might be “Meg takes 3 apples from Sally”. If we wrote it with math symbols we might have $M+3$ and $S-3$ going backwards.

When you and your partner agree on a description, write it down.

If you haven't already, take a number or result from the end of the problem and use it in your reverse “script” to work backwards and solve the problem.

Key Outcomes

- Change the operations in the problem into their inverses.
- Take an ending number and undo what was done to it to get back to the beginning
- Figure out other ways to describe the problem so that it can be worked backwards more easily.
- Develop a plan by comparing and talking through ideas with a partner.
- Solve the problem.

III. *Checking your Answer* (No Algebra)

Format: Each pair joins with another pair or two to form groups of 4-6.

Multiple strategies are presented below for checking the answer. Students should complete as many as possible.

Sample Activity:

Comparing: Did each pair have the same steps or different? Do the differences change the answer? Did each pair use the same inverse operations? If not, did this change the answer?

Trying the Script: Swap your reverse scripts and each pair should try to follow exactly what the other pair's script says. Do you get the same result? If not, see if you can come to agreement.

Generalizing: Did you figure out a pattern or rule you could follow to solve the problem again with different numbers or another version of the problem? What happens when you check your answer using this pattern or rule?

Key Outcomes:

- Use different strategies to check the solution.
- Retry the problem forwards.

IV. *Checking your Answer* (Algebraically)

Format: Each pair joins with another pair or two to form groups of 4-6.

There are many important relationships between working backwards and solving problems algebraically. As you wrote your reverse scripts and worked through the problem backwards, the quantities and the relationships in the problem probably became more obvious. The following activities are designed to help you notice and use these relationships.

Sample Activity:

- 1) If you haven't already, assign variables to the unknowns in your scripts for the problem. Go through the steps of the problem, trying to represent them with algebraic expressions.
- 2) By the time you get to the end of the problem scenario, you might have algebraic equations that you can plug the ending numbers of the problem into and solve.
- 3) Compare your problem solving with the algebra and with the working backwards process. Did you get the same answer? Did the algebra process help you catch errors from the working backwards method? Did the working backwards solution help you find errors in your algebra? Did you discover any new information or relationships?

Key Outcomes:

- Make connections between the “undoing” or inverting processes involved in working backwards and solving an algebraic equation.
- Use the working backwards process to help you understand the quantities and relationships in the problem so that you can write an equation.
- Use algebra to check the results of working backwards, and use working backwards to make sure your algebraic steps make sense.