The Math Forum: Problems of the Week

Problem Solving and Communication Activity Series

Round 10: Cases

Case-based reasoning helps problem solvers to understand the problem, work towards a solution, surface interesting mathematics, and verify the robustness of their solutions. To understand the problem, problem solvers might test interesting or representative cases, and think about the different outcomes they see. When solving the problem, they might use cases to consider when certain outcomes will occur, or to narrow down the possibilities they have to investigate. Some problems have different answers for different cases. Exploring different cases can lead to questions that problem solvers might explore further, like, "what would happen if I used a negative number?" or, "would this work for obtuse triangles, too?" Finally, when determining whether a possible solution is correct, good problem solvers test their solution using multiple cases, especially cases that they know behave differently.

In this document, we offer three activities. The first, "Possible Cases," helps students surface different cases that might be critical. The second, "Implications" helps students use cases to understand the problem and use cases to work towards a solution. The third, "Accountability," encourages the students to determine as a large group what the interesting cases in the problem were, and confirm that their answer addresses all of those cases.

The activities are written so that you can use them with problems of your choosing.

Problem-Solving Goals

Breaking problems into cases can help problem solvers:

- Be explicit about types of objects in the problem, specifically those that students often forget about (negative numbers, straight angles, etc.)
- Be aware of the many ways that problems can be broken into cases.
- Understand the problem by testing representative objects from multiple cases.
- Analyze the behavior of the mathematics in the problem in different cases.
- Use cases to check their solutions.

Communication Goals

Students that attempt to break a problem into cases can use the writing process to:

- Record possible cases, and what happens mathematically in different cases.
- Keep track of different situations they have to check.
- Share ideas about what is relevant in the problem, and come to a group consensus about it.
- Develop their reasoning and move from explanation to proof.

Note: There is an implicit assumption throughout this *Activity Series* that users know that they can also be making use of strategies that have been introduced in prior weeks. In particular, *Understanding the Problem* was developed as a set of strategies that is always useful and to some extent assumed to be in use, even when focusing on a new strategy, such as *Cases*.

Activities

I. Possible Cases

Format: Students working individually or in pairs, then sharing with a larger group.

The first step to breaking a problem into cases is to identify the important objects used in the problem or the different situations to consider in the problem. What is the problem about? What are the countable or measurable quantities that can take on different kinds of values?

Sample Activity

Work individually or in pairs to begin filling in the blanks of the following prompts for just a few minutes. Then share ideas with the larger group of 3-4 students. At this point, you should be focusing on brainstorming and generating as many ideas as you can for breaking the problem into different situations. Don't solve the problem yet.

Fill in as many of the blanks as you can in the following *Cases* sentences:

1) The mathematical objects or quantities (things I can count or measure) used in this problem are _____.

2) The different types of mathematical objects or quantities I might want to think about are _____.

   OR

   Some different types of situations are _____.

Here are some examples of answers to these sentences:

- Numbers: odd/even, positive/negative, integers/fractions, multiples of a certain number, factors of a certain number, etc.
- Angles: acute, right, obtuse, etc.
- Quadrilaterals: trapezoids, parallelograms, rhombi, rectangles, squares, etc.
- Situations: maximize one quantity in the problem, consider different values for a certain quantity in the problem, etc.

It is a good idea when brainstorming in the group to write down the ideas stimulated by hearing from each other. After brainstorming, discuss which types of cases you think are most promising or interesting.

Key Outcomes

- Identify objects or quantities that are relevant to the problem and solution (variables or inputs).
- Be aware of the different cases that do (and don't) exist for the problem.
- Explicitly record those cases in order to be accountable for them throughout the solution process.

II. Implications.

Format: Students working with partners.

One way to understand a problem better is to think about which cases will make a difference in the way the problem works out. Once you have selected some cases to look at, it is important to try them and think about whether you would get the same result for any object/quantity of that type. For example, if you are testing odd numbers, you want to think about whether something different happens for any odd number than happens for any even number. If you are thinking about angles, you want to see if acute angles create a different situation from obtuse angles in your problem.

Sample Activity

Step 1: With your partner, select a case that you would like to test out and explore. Answer the question "What happens if?" for that case. For example: what happens if you use a positive number? What happens if the angle is smaller than 90 degrees? Each partner should pick a different example to test, and see what happens in the problem.

Step 2: Compare your results. a) What did you learn about the problem? b) What did you learn about that case?

Step 3: If you have enough information to solve the problem, do that. Otherwise, pick another case to think about in order to solve the problem.

Key Outcomes

- Analyze which types of cases will be useful and which don't matter as much.
- Understand more about the mathematical relationships in the problem.
- Be aware that, especially when working with numbers, cases might be very specific (e.g. numbers greater than 4 but less than 10).

III. Accountability

Format: Large group brainstorming moving to small-group or pair work.

One very important way that mathematicians use cases is to make sure that their proposed solutions to problems are accountable to all possible types of inputs. They try to identify all the relevant cases, and test their solution for each case, to make sure they're absolutely right.

Sample Activity:

Step 1: Once your pair has found a solution or come to a good stopping place, write down the most useful cases you tried, and what you learned from them. Write on a big sheet of newsprint or in some place where the class can look at your thinking about cases. What did you learn from each case? What did you learn about the whole problem?

Step 2: Hang your poster up around the room. Walk around and look at other posters. If two or more groups tried similar cases, you might move the posters so that they are next to one another.

Step 3: As a class, ask clarifying questions about any of the posters.

Step 4: Share what you noticed as you walked around the room. Similarities? Interesting insights? Different results?

Step 5: What are you still wondering about? What other issues do you need to resolve before you have a final answer?

Step 6: Back with your partner(s), write up your final answer (for submitting to the AlgPoW, we hope!)

Key Outcomes:

- As mathematicians do, come to a group consensus about what is relevant to the problem at hand.
- Also as mathematicians do, share the work of testing and calculating, and evaluate results based on the group consensus.