The Art of Asking Thought Provoking Questions in the Mathematics Classroom

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The current research about how students learn shows that it is important for every educator to cultivate the art of asking questions. Textbooks are filled with conceptually low level questions requiring rote memory or simple calculations. However, questions that lead to learning with understanding are not readily found in textbooks. Such questions are not to be confused with exercises that require clever applications of *previously learned* skills. Invariably, exercises in textbooks represent what the author expects the students to be able to do because his abundantly clear explanation has endowed them with the necessary *understanding*. Textbook exercises are quite useful, but no text can provide a universal series of questions that provoke every student or even every class to think through a new concept. Questions that lead students to *understand* must start at their current understanding and provoke them to think forward to the mathematics at hand. Questions of this type are valuable instructional tools, but textbooks are not designed to provide them. Good, make-them-think questions take practice to design and use effectively.

Before deciding that this is for somebody else's students or somebody who has more time, ask yourself when was the last time you heard "these students don't even know the basics" or "they can't even read the easy problems much less solve them" or "they act like they are bored, but they really don't get it" or "there is no way my students can do those problems!" While such statements, made in private of course, may seem like harmless venting of frustration, they are formidable obstacles to students really understanding the mathematics we purport to *teach*. Convincing ourselves that students *can't get it* dissuades us from challenging them to really think about mathematics. So, many of us limit ourselves to training students to execute *procedures*. Substituting in a formula or mechanically executing some procedure is not mathematics, as Mark Twain would say, it is French. In order to learn and understand mathematics, students must think about mathematics. Our job is to *provoke* that thinking. To that end, we propose integrating conceptual questions into the classroom routine. Conceptual questions are those that ask the student to think deeper about what they are doing, why what they are doing works or not, when what they are doing will work and when it will not, and the relative efficiency of what they do to solve a problem. In this context, the primary purpose of conceptual questions is to provoke learning not to ascertain what has already been learned.

Creating and using questions that effectively provoke thinking and learning requires patience and practice. Like most change, first efforts to provoke students to think on their own will meet resistance. Initially some students will evade the purpose and seek ways to short cut the process. Careful phrasing and opportune timing are needed to capture and focus their attention. As students realize (a little promotion helps here) that understanding is the key to real success, they become more willing to participate and even contribute to the process. Students like succeeding as much as anyone. Fortunately, even from the beginning it is not necessary to create perfect conceptual questions from scratch. The lesson of the day can be focused on learning with understanding through by adapting and stretching one or two routine textbook exercises.

Going beyond the *right answer*

Sometimes the toughest part of effectively using questions to provoke student thinking is listening to their answers, especially for experts. We experts already know the *right answer* and an efficient way to get to that answer, so waiting patiently while students take blind alleys and circuitous routes can be exasperating. This is especially true in mathematics where the accepted benchmark is the *right answer*. We like *right answers*, we listen for them, and we're ready to move on when we hear them. Of course, when a student does not get the *right answer*, we *know* to look for a mistake or misunderstanding. We may or may not take the time to do so. Conversely, if a student gets the *right answer*, then we happily *assume* that the student understands and look no further. So when a student simplifies $\frac{16}{64}$ to $\frac{1}{4}$, we assume that he has found the common factor of 16 and canceled. If the student mistakenly canceled the 6s, that is $\frac{1\cancel{6}}{\cancel{6}4}$, saying *right* and moving on to the next question reinforces a malicious misconception. While it is rare that a student will make this particular mistake, canceling the $2x^2$s in the expression $\frac{3x^4 - 2x^2}{x^3 - 2x^2}$ to get 3x is a common error that has its roots in memorizing procedures without understanding. In the mathematics classroom, we rarely look beyond the answer. This means that instead of catching the opportunity to counteract a critical misunderstanding when it happens, we either say *that's the wrong answer* or worse *that's right*.

When we look only at the answer, we can't be sure what a student is thinking, how he got that answer or why she believes it is correct. Whether the answer is correct or not, the only way to discover how a student got to the answer is to ask for an explanation. Asking students to explain how they get an answer or to show why their answer has to be correct pushes them to think more about the mathematics involved, regardless of the procedure used. This kind of question shifts the emphasis from the right answer to the reasoning behind the answer which is where the real mathematics lies. Even if the point is to learn some basic arithmetic skills, current learning research shows that students who reflect on their own thinking processes develop more effective learning habits.

There are several ways of pushing students beyond the answer and they need to be pushed, at least in the beginning. Ask the student to:

- explain the reasoning behind the solution of a problem,
- explain the strategy used to solve a problem,
- justify answers and/or choices,
- explain what the answer means in a particular context,
- predict what will happen next,
- recognize and understand questions stated in a novel form, or
- state a question or invent a problem for which the answer is given.

Adapting textbook exercises

In mathematics, mastering basic skills and procedures is quintessential. So, most of the questions in textbooks (and on our exams) require mechanical execution of standard procedures. Sometimes we forget that students have neither our experience nor our perspective to be able to appreciate when and where these skills and procedures will prove their value. Our emphatic 'this stuff is really important' is less than adequate motivation. It is more effective to show by example that mathematicians value the reasoning more than the procedures or formulas. After all, it is the cumulative reasoning of many mathematicians over many years that led to procedures that are remarkably effective for solving problems. Besides asking how and why in order to focus on the mathematics behind a procedure, ordinary run-of-the-mill textbook questions can also be modified to target a higher cognitive level. Some examples of particular opportunities to do this follow.

Common errors

Identify a common mistake in student work and modify a question to confront it.

Routine question #1 often results in the common error of adding discounts.

> Some stores offer an additional discount on items that have already been marked down. There is a 25% discount on a blouse that was originally priced at $60. If the store offers an additional 15% discount, what is the final price of the blouse?
>
> Even though it is incorrect, many students add 25% and 15% to get 40% and expect a total discount of $24 for a final price of $36.

Modified question #1 helps students confront the common error of adding discounts.

> Some stores offer an additional discount on items that have already been marked down. There is a 25% discount on a blouse that was originally priced at $60. Today the store is offering an additional 15% discount on everything that has already been marked down. Juanita figures that discounts of 25% and 15% add up to 40%, so that should give her a $24 reduction on the $60 blouse for a final price of $36. When she takes the blouse to the cashier to pay for it, the final price is actually $38.25 (without sales tax). Explain to Juanita why the final price is $38.25 and not $36.

Routine question #2 frequently results in the common error of averaging averages.

> If the average velocity of a car during the first 50 miles of a 100 mile trip was 40 mph and the average velocity for the last 50 miles was 60 mph, what was the average velocity for the 100 mile trip?

Even though it is incorrect, many students average the two average velocities to get an average velocity of 50 mph for the entire trip.

Modified question #2, or rather in this case a series of questions, helps students confront this common error.

If the average velocity of a car during the first 50 miles of a trip was 40 mph, how long did this part of the trip take?

If the average velocity of a car during the last 50 miles of a trip was 60 mph. how long did this part of the trip take?

If the average velocity of a car during the first 50 miles of a trip was 40 mph and the average velocity during the last 50 miles of the trip was 60 mph, how long did the entire 100 mile trip take?

Explain why the average velocity of the 100 mile trip in the previous problem was not 50 mph, the average of the two average velocities.

Suppose a car was driven at 40 mph during the first part of a 100 mile trip and at 60 mph for the remainder of the trip. If the average velocity for the entire 100 mile trip was 50 mph, for how many miles was it driven at 60 mph?

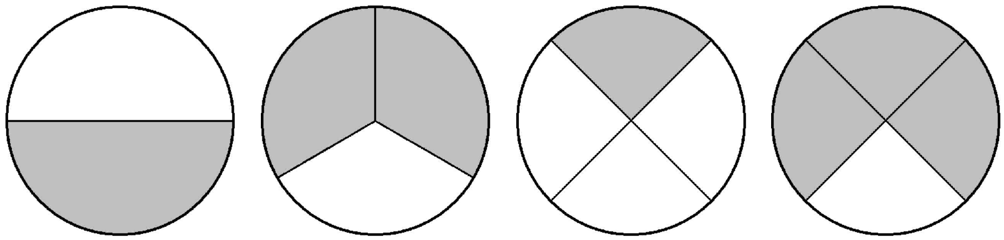
It is quite common to suppose that the average velocity during a trip that has two parts is the average of the average velocities during those two parts. Under what conditions is this true and under what conditions is it false?

Adapting a rote-memory question

Questions that students answer using some memorized procedure do little to provoke deeper thinking on their part. Higher level thinking can be stimulated by asking the students to look at the concept behind the original question in a different way, to interpret the answer, or to extrapolate to a new context.

Routine question #3 asks students to identify a pictorial representation of a fraction.

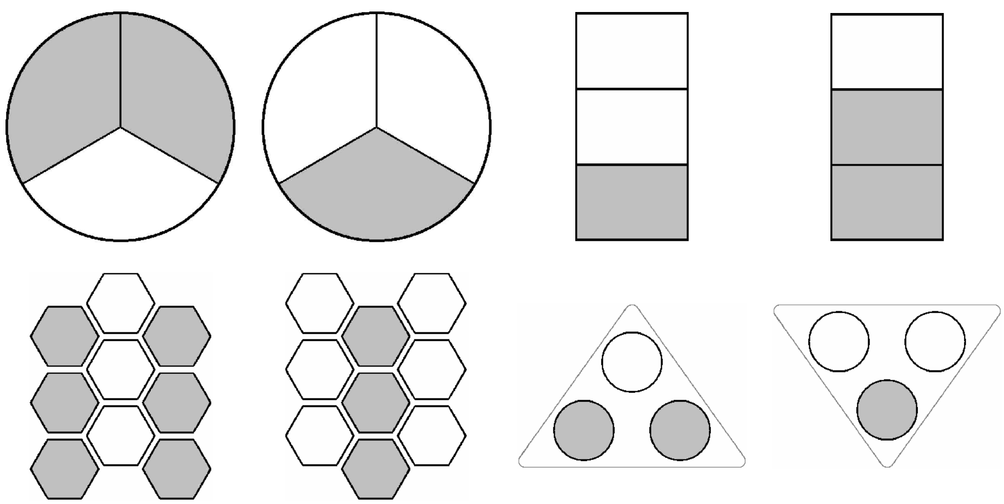
Which of the following figures represents 2/3?



This question asks the student choose, in a single specific context, the shaded portion of a circle that represents 2/3. The student who answers correctly reveals little about his understanding of fractions.

Modified question #3 asks students to interpret how a picture represents a particular fraction.

Explain how each of the following figures represents the fraction 2/3.



The figures in this question represent 2/3 in slightly different contexts (in the case of the large circles and the rectangles as a part of a whole and in the case of the hexagons and the small circles as parts of a group). Asking for an explanation of how each of the figures represents 2/3 helps the student realize that fractions have different meanings in different contexts. From an assessment or information for the teacher perspective, the answer reveals a student's understanding of fractions and her ability to interpret them in multiple contexts.

Routine question #4 asks students to count the number of possible combinations in a particular situation.

> If there are seven different colored marbles in a bag and three marbles are randomly drawn from the bag, how many different combinations of three colors are possible?

Asking for the number of possible combinations of 7 objects taken 3 at a time could be a challenging question. However, in most cases, this question is asked after the student has been given a formula to calculate C(n,r). In this case using the formula is a modest application, but it does not require much higher order thinking to substitute in the formula and perform the arithmetic.

Modified question #4 asks students to make an application of combinatorial counting in probability.

> There are seven marbles in a bag. Five of the marbles are red and two are blue. If three marbles are drawn randomly from the bag, what is the probability that all three marbles will be red?

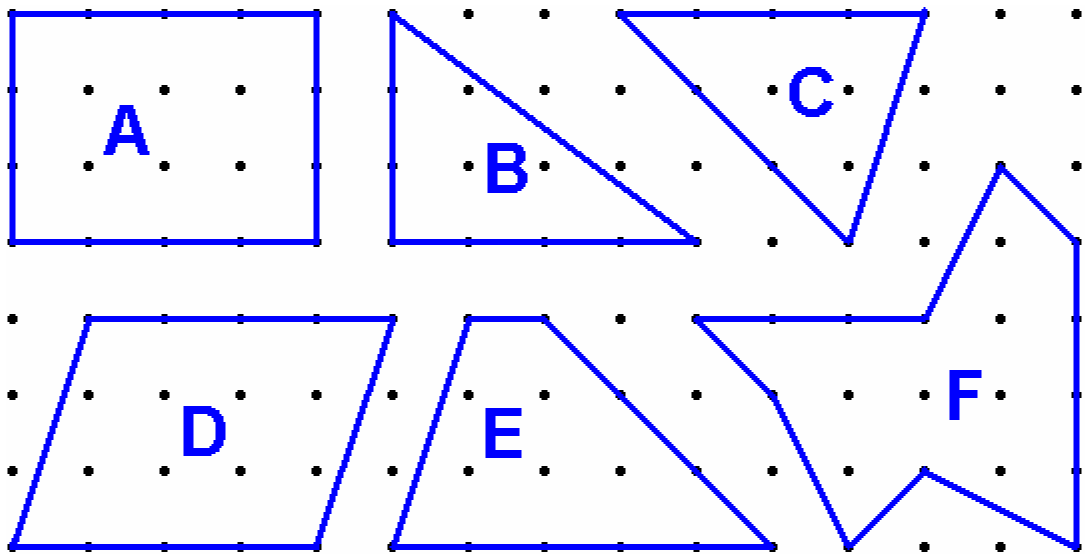
Now, if the student knows the formula for calculating combinations, this question may require applying it to a novel situation. The novelty depends on what has been previously taught about probability. Even though the same formula is useful for counting all of the relevant possibilities, the student first must understand that it can be applied to count the combinations that correspond to three red marbles.

Modifying a basic arithmetic question

A basic arithmetic question can be modified to ask the student for a qualitative answer in which he demonstrates application, analysis or synthesis.

Routine question #5 asks the student to find the area of various triangles and cuadrilaterals.

> If the shortest distance between two dots in the following figure is one centimeter, find the areas of figures A, B, C, D, and E.



Although a student could find the areas of figures A through E by counting the square centimeters inside the figure, most students will resort to a variety of formulas. Whether they already know the formulas or they use some reference to find the formula, finding the areas of these figures quickly be reduced to basic arithmetic.

There are some interesting questions that can be asked about the areas of these figures.

Modified question #5A asks the student to find the area of an irregular figure.

> If the shortest distance between two dots in the figure shown above is one centimeter, find the area of figure F.

In order to find the area of figure F, the student can break the figure into simpler pieces and use familiar formulas for area. If so, the student shows an understanding of area beyond substituting in formulas. If this is the case, asking for the area of figure F would be a more conceptual question. However, if the student has been trained to cut figures into familiar parts, then finding the area of figure F is little more than execution of a memorized procedure.

Modified question #5B asks the student to relate the areas of distinct shapes.

> Explain why the area of figure D has to be the same as the area of figure A no matter what the shortest distance is between two dots in the diagram.

Modified question #5C asks the student to demonstrate that a relationship holds between the areas of clearly related shapes.

> Explain why the area of figure B has to be exactly half of the area of figure A no matter what the shortest distance is between two dots in the diagram.

Modified question #5D asks the student to demonstrate that the areas of two dissimilar shapes must be the same.

> Explain why the area of figure B has to be the same as the area of figure C no matter what the shortest distance is between two dots in the diagram.

Making the explanations called for in the previous three questions will reveal something of the student's understanding of area, no matter what means they use. However, with the added modification of *Use drawings to explain . . .*, the answers can reveal still more about the student's understanding of area. In particular, an explanation of #5D with drawings requires the student to analyze the relationship between figures B, C and A.

Modified question #5E asks the students to find and explain the relationship between two dissimilar shapes.

> If the shortest distance between two dots in the previous diagram is one centimeter, what is the relationship between the areas of figure E and figure C? Explain.

In order to answer this last question the student must decompose figure E into parts that correspond to figure C and a parallelogram or components of figure C and a rectangle. This decomposition or visualization strategy is valuable in and of itself. After giving plenty of time to work through the problem, providing still more time to have students reflect on and discuss what was the most effective strategy for solving the problem will help them make it their own. One technique to do this is to ask students to put the strategy in their own words. Now, they may write something like - *break it into different pieces that we know how to find the area of and then add the areas*. This falls a little short of *break something down into its components and examine the parts in order to get a better* *understanding of the whole*, so more problems and discussions will be needed over the course of time.

Modified question #5F asks the students to discover an important mathematical theorem.

> If the shortest distance between two dots in the diagram above is one centimeter, find the relationship between the number of dots in the interior and on the boundary of the figure and the area of the figure.

The last question requires the student to synthesize a formula for the area of lattice polygons, better known as Pick's theorem. This kind of question gives the students an opportunity to do mathematics instead of just using mathematics. Of course, the students will need an adequate amount of time to work with something like this. They may need some scaffolding – say preparing a table with the number of interior dots, the number of exterior dots, and the area – in order to find the formula, but with a little coaxing they can come up with the scaffold themselves. After discovering the formula, the students can and should be asked why it works. This provides a great opportunity to point out that their discovery is a *generalization*, one of the really big ideas in mathematics, while the 'why it works' is another called justification.

Ask for a Qualitative Answer

A rote-memory or basic arithmetic question can be modified to ask for a qualitative answer; for example, a list of increasing or decreasing properties, "greater than", "less than", or "no change", always with explanation required.

Routine question #6 asks the student to perform a basic arithmetic operation with fractions.

$$\frac{2}{3} \div \frac{3}{5} =$$

Generally, students have been trained to *invert the divisor and multiply*. There is a common, persistent belief that multiplying makes numbers bigger and dividing makes numbers smaller. This comes from experience with multiplication and division of positive integers. In fact, they may have been *taught* this in elementary school.

Modified question #6A asks the student to analyze this belief.

The final answer for $\frac{2}{3} \div \frac{p}{q}$ is larger when

a) $p = 3$ and $q = 5$

b) $p = 3$ and $q = 4$

c) $p = 4$ and $q = 5$

d) $p = 4$ and $q = 4$.

The purpose of this question is to get the student to analyze the effect of changing the numerator or the denominator of a positive divisor. Even if the student actually does the arithmetic for each of the alternatives, she will at least have to compare the results. Even more directed, deliberative questions may be used to get the students to look carefully at the effect of changing the numerator or the denominator when dividing, such as one of the following.

Modified question #6B asks the student to compare the effects of increasing the numerator of the divisor.

Suppose p and q are positive numbers, does the result of the division $\frac{2}{3} \div \frac{p}{q}$ increase or decrease if the value of p increases? Explain why.

Modified question #6C asks the student to compare the effects of increasing the denominator of the divisor.

Suppose p and q are positive numbers, does the result of the division $\frac{2}{3} \div \frac{p}{q}$ increase or decrease if the value of q increases? Explain why.

Answering both modified questions #6B and #6C helps the students confront the common expectation that multiplication makes a number bigger and division makes it smaller.

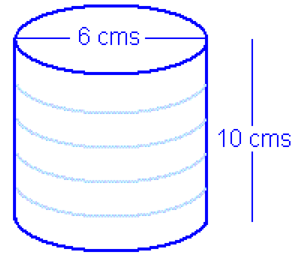
<u>Modified question #6D</u> asks the student to apply the implications of the answers of #6B and #6C.

> The result of the division $\frac{2}{3} \div \frac{p}{q}$ increases when the value of p decreases by 1 and when the value of q increase by 1. If the result of the division $\frac{2}{3} \div \frac{p}{q}$ increases more when the value of p decreases by 1 than when the value of q increases by 1, what can you say about the values of p and q? Explain in detail.

If the student understands that dividing a positive number by a positive number means separating in that size parts, he can answer the first two modified questions with a minimum of work. He only has to figure out when the divisor increases. In order to respond to the last question, the student needs to understand that the largest result corresponds to the smallest divisor. After examining several examples, a student may *guess* that p has to be smaller than q. However, a clear explanation why this condition is necessary and sufficient requires comparing the two divisors $\frac{p-1}{q}$ and $\frac{p}{q+1}$. In effect, the student has to solve the inequality $\frac{p-1}{q} < \frac{p}{q+1}$. Answering modified question #26D will take time, but answering this question stretches the student's understanding of division by a positive fraction and confronting the belief that division makes numbers smaller can help the student avoid future misapplications of the *invert and multiply rule*. To stretch this understanding a bit more, a good follow up is to ask how the answer changes if p and q are not required to be positive.

<u>Routine question #7</u> asks the students to calculate the volume of a cylinder.

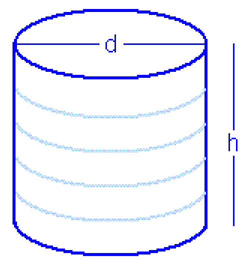
The diameter of the base of a can is 6 centimeters and the diameter is 10 centimeters. Find the volume of the can.



To answer this question the student finds the radius of the base and substitutes the values for the radius and the height in the standard formula for the volume of a cylinder. This is very direct, but it does not confront the common belief among both teachers and students that doubling all of the dimensions doubles the volume. That is, doubling the radius and volume will have the effect of doubling the volume.

Modified question #7A asks the student to compare the effect of doubling the diameter with the effect of doubling the height.

The formula for the volume of a cylinder is $\pi r^2 h$ where r represents the radius of the base and h represents the height. Which of the following will double the volume?



a) double the diameter

b) double the height

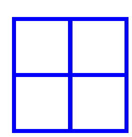
c) double the diameter or double the height has the same effect

d) both the diameter and the height have to be doubled to double the volume.

This question is more a starting place for the teacher to ascertain how well the students understand the relationships between linear measures, area and volume. Depending on the answers, more questions may be needed. The misconception that area and volume are linear is very common and stems in large part from memorizing formulas without developing the underlying understanding. One way to follow up on the previous question is to ask for explanations with drawings and diagrams.

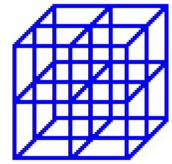
Modified question #7B asks the student to relate a drawing and an algebraic expression.

Use the diagram at right to explain why $(2x)^2$ must be the same as $4x^2$.



Modified question #7C asks the student to relate a two dimensional drawing of a three dimensional object and the relative volumes of two objects.

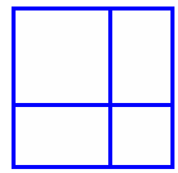
Use the diagram at right to explain the effect of doubling the sides of a cube on the volume of the cube.



The two previous questions provide the diagrams. It requires more time, but asking the students to make their own diagrams would be even better.

Modified question #7D asks the student to confront a common algebraic error.

Use the diagram at right to explain why $(a + b)^2 \neq a^2 + b^2$ unless a = 0 or b = 0.



Again the same question without the diagram, would require more time and would probably be more effective.

Invert a Calculation

Instead of giving the students a calculation to do, give them the answer and ask them to invent the problem that has that answer. An alternate form of this is to ask the students to prepare (without the textbook of course) questions for their test.

Routine question #8 asks the students to add two numbers.

7 + 3 =

Modified question #8 gives students the answer and asks for the problem.

Find two numbers whose sum is ten.

The previous question is particularly useful when students are learning to add one digit numbers. Thinking about pairs of digits that sum to ten can help students become efficient adding columns of numbers and later adding two-digit numbers.

Routine question #9 asks the students to interpret a situation and divide by a fraction.

If three acres are divided into quarter acre lots, how many lots will there be?

This arithmetic involved in this question is similar to #6. Even though students have been *trained* to *invert the divisor and multiply*, they often do not relate this procedure to the understanding they bring from the division of integers. As a result, some students and even some teachers lack confidence when their answers are larger than the dividend.

<u>Modified question #9A</u> gives the student the result of the division of a large lot into smaller lots and asks for the size of the original lot.

> If a large lot is divided into twelve quarter acre lots, how large was the original lot? Explain your answer.

Inverting the calculation in this case helps the student examine the relationship between division and multiplication. Though oft repeated, *division is the inverse of multiplication*, students need to develop an understanding of what this phrase means. The explanation is a key part of the question. Actually, it would be even better to present both questions and asks for diagrams to illustrate each case.

<u>Modified question #9B</u> asks the student to represent the situation in two different ways.

> A three acre lot has been divided into twelve quarter acre lots. Draw a diagram that represents this. Write an expression that represents the number of small lots as the result of an operation involving the size of the original lot and the size of the smaller lots.

<u>Modified question #9C</u> asks the student to represent the situation in two different ways.

> A large lot has been divided into twelve smaller lots that are one third acre each. Draw a diagram that represents this. Write an expression that represents the size of the original lot as the result of an operation involving the number and the size of the smaller lots.

<u>Routine question #10</u> asks the student to solve a quadratic equation.

> Solve the quadratic equation $2x^2 - x = 10$.

The student must change the original equation to standard form and then factor or apply the quadratic formula. Since the equation is relatively simple and the roots are rational, the student is expected to factor and solve. Even after factoring to solve the equation, the student does not necessarily recognize the relationship between the linear factors and the roots.

Modified question #10A asks the student to invent a quadratic equation with specified roots.

> Write a quadratic equation with integer coefficients so that 2.5 and -2 are the solutions.

Constructing the equation with the specified solutions, the student implicitly uses the connection between roots of an equation and linear factors.

Modified question #10B asks the students to make this connection more explicit.

> How you would make quadratic equation with two specified solutions a and b? How do you know that your equation will work?

Extra Data

Most textbook exercises include exactly the numerical information required to solve the problem. This leads to the common strategy of figuring out where the supplied numbers fit in the procedure of the day. Such arithmetic mumbo-jumbo is not effective on the final exam, much less in the real world where problems rarely present themselves with just the data required for their solution. Introducing extraneous data or presenting the data in an unusual format increases the level of conceptual engagement.

Routine question #11 provides exactly the data needed to find the answer.

> If the neighborhood station sells gasoline at $2.469 per gallon, how many gallons of gas can Pablo put in the tank of his car for $25?

As stated the question can be answered by performing division. There is little chance that the student with any familiarity with gas mileage will invert the divisor and the dividend.

Modified question #11 provides several related numerical values.

> Pablo drives a 2003 Montero that gets 18 miles per gallon. At the station where Pablo buys gasoline, regular sells for $2.469 per gallon. How many gallons of gasoline can Pablo buy with $25?

The latter version of the question requires a little more thinking on the part of the student. Asking the student to explain how she arrived at her answer could require a still higher cognitive level of thinking and at least would reveal her thinking.

Routine Question #12 poses a familiar word problem.

> Juan was 14 years old when his little sister Rosita was born. How old will Rosita be on Juan's 24th birthday?

Even though the students really need to think the question through carefully to come to the correct answer of nine years old, many will not. Instead of persuading the student to think, this question reinforces a common notion that mathematics is a collection of clever tricks. To be precise, a complete answer would note that Rosita will be nine years old unless Juan and Rosita have the same birthday, in which case Rosita will celebrate her tenth birthday on the day that Juan celebrates his twenty fourth. While the complete answer requires the sort of attention to details that typifies mathematics, the question does little to provoke the students to think about these details.

Modified question #12 asks a similar question but includes some extraneous information that can cue the student to consider the relevant details.

> Juan was fourteen years old when his little sister Rosita was born. Juan married María on August 12, a month before his 25^{th} birthday. If Rosita's birthday is January 6, how old was Rosita on Juan's fiftieth birthday?

In our endeavor to prepare citizens to face the challenges that await them, it is important that they can learn new technologies and adapt to new situations quickly. The model of preparing people to be productive, responsible citizens by training them to understand and follow instructions is hopelessly archaic. Today's society needs a creative population that can analyze new problems and find new solutions to old ones. The idea that mastery of mathematics means knowing a lot of formulas and being quick at arithmetic was never adequate. At least for this century, mastery of mathematics means being able to adapt known procedures to new situations and come up with more efficient procedures for old situations. Recent research in how people learn mathematics shows that understanding what they are doing and why is critical. Such learning does not happen spontaneously, someone has to provoke the student to do the thinking required to understand what they are doing. We challenge you to provoke the students in your classroom to think, to do, and thereby to learn mathematics by posing more conceptual questions and listening for their answers.

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