Geoboards in the Classroom

Tom Scavo

August 29, 1995

To the teacher

This unit deals with the length and area of two-dimensional geometric figures using the geoboard as a pedagogical device. Even though your students may have already seen some of this material, it is still likely that many of them have only a superficial understanding of length and area. It's quite amazing, for example, how many students have trouble distinguishing between perimeter and area—even secondary students confuse the two.

A typical student's understanding of units is even more problematic. Ask your students to describe a square inch, for example. It's unfortunate, but true, that many write "in^2" without really understanding what it means. But with a geoboard, a student can *see* the area of a rectangle, say, and will readily count the square units making up such a geometric figure. Indeed, "square unit" takes on a whole new meaning with the geoboard.

Materials

Geoboards come in all shapes and sizes. These lessons, in particular, assume a square 5×5 geoboard with a total of 25 pegs (see Figure 1), but other configurations are also possible. It is advisable, but not essential, that the geoboards are constructed in such a way that four of them fit together to make a square 10×10 board with 100 pegs as shown in Figure 2. This desirable feature permits extended activities and gives the students additional opportunities to discover geometric and algebraic patterns. The ideal geoboard will also have a circular configuration on its reverse side.

Unfortunately, few commercial geoboards have these specifications. On the other hand, students can build their own. All that's needed is a square piece of hardwood (with beveled edges) and 25 brass pegs (the kind that don't pull out). It's fun and educational to design and build an aesthetically pleasing and functional geoboard. If you find that such a project is not

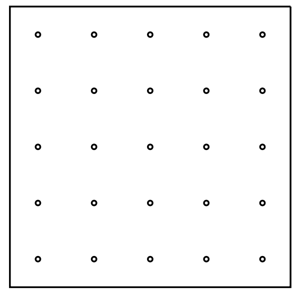


Figure 1: A square 5×5 geoboard

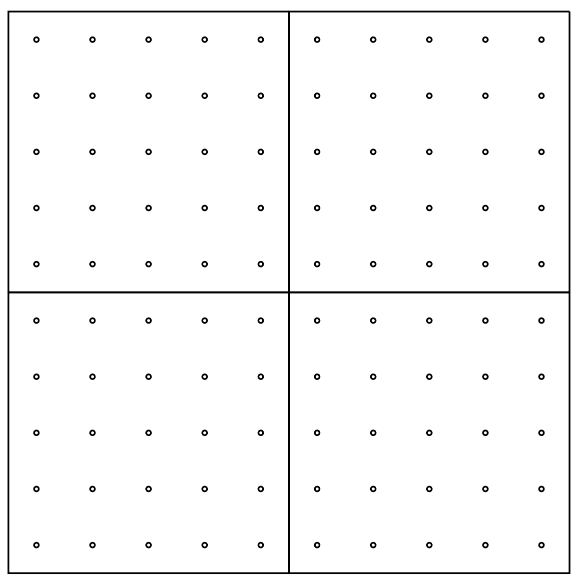


Figure 2: Four 5×5 geoboards fit together to make a 10×10 geoboard

feasible, however, have your students construct paper models instead. Either way, determining the proper spacing of the pegs (so that four geoboards fit together to make a larger geoboard) can be a rewarding and challenging exercise.

A large number of rubber bands—all sizes and colors—will be needed for these activities. Each student must have a generous supply of rubber bands. Be aware of potential problems, however. Most teachers will want to make it very clear at the beginning of the unit that rubber bands are to be used for geoboard activities only. All other uses are inappropriate and not allowed. A written contract to this effect sometimes helps students to take this important rule seriously.

Finally, it's a good idea for students to record their geoboard activities on paper. Dot paper serves this purpose well (see Appendix A) and should be freely available. Each student will also need a pencil, eraser, and ruler.

Activities

These lessons should be introduced into the classroom in numerical order. It's not necessary to complete every activity in each lesson, but at least some of the concepts should be covered before moving on to the next lesson. Indeed, the activities have been purposely designed for a broad range of students so that teacher and student may choose an appropriate subset. We anticipate that students will revisit the lessons in more depth in subsequent years.

Length

Determining the length of a horizontal or vertical line segment on the geoboard is easy; even primary students can do it—by counting. A diagonal line segment is more difficult to measure, however. Many students are content to estimate the length of diagonal line segments, which should be encouraged. More mathematically sophisticated students will want to compute such lengths directly.

One calculates the length of an arbitrary line segment using the Pythagorean theorem, a concept whose understanding is a primary objective of this unit.[1] As a teacher, you must be able to apply the Pythagorean theorem on

[1]We are not suggesting that first graders be taught the Pythagorean theorem. The

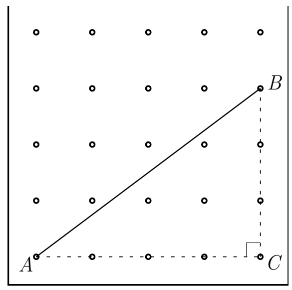


Figure 3: Computing the length of a diagonal line segment

sight, since students are surely going to ask you questions regarding length early in the unit.

Here's how it's done. For any given line segment, construct a right triangle having that line segment as hypotenuse. For example, to calculate the length of line segment AB in Figure 3, construct the right triangle ABC. The base of this triangle (line segment AC) is 4 units long, whereas the height (line segment BC) is 3 units. Using the Pythagorean theorem, the length of line segment AB is found by calculating *the square root of the sum of the squares* of the lengths of the base and height of the triangle, that is,

$$\sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

Thus the length of line segment AB is 5 units. Of course, the sum of the squares is not always a perfect square (in fact, this will seldom be the case), so the length will be an irrational number,[2] in general. Not all students will be ready for square roots, but the teacher is advised to know how to do such calculations nonetheless.

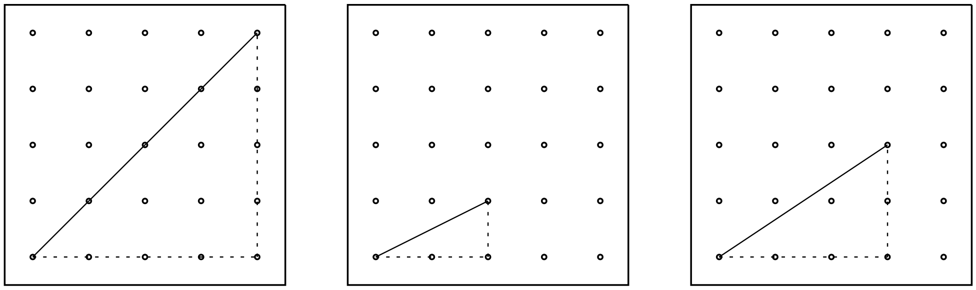
Additional examples of the Pythagorean theorem are given in Figure 4.

Area

Many students will already know how to calculate the area of squares and rectangles. Some will also know how to compute the area of parallelograms and triangles. The geoboard will help these students strengthen their understanding of area even more. But rather than give them formulas to calculate

lessons provided will almost certainly be spread out over many years, starting with the elementary grades on up to middle school or even high school.

[2]An irrational number is not a rational number (or fraction).



(a) $\sqrt{4^2+4^2} = \sqrt{16+16} = \sqrt{32}$ (b) $\sqrt{2^2+1^2} = \sqrt{4+1} = \sqrt{5}$ (c) $\sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$

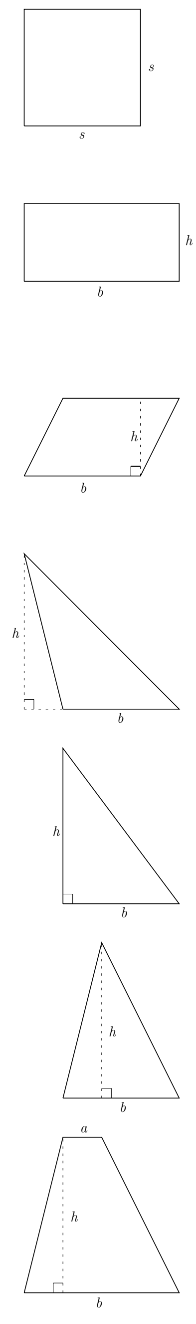
Figure 4: More examples of length calculations on the geoboard

area, we will try to give them conceptual tools that tend to persist in their minds long after the formulas are gone.

For example, if a student were to take a pair of scissors and snip a parallelogram along its height, she would quickly realize (by fitting together the two pieces) that the area of a parallelogram is nothing more than the area of the corresponding rectangle (see Figure ??). Similarly, a right triangle may be thought of as half a rectangle as depicted in Figure ??. These kinds of associations help students understand area much better, and the geoboard is an excellent device on which to illustrate these concepts.

Still, the teacher must be able to calculate the area of certain polygons on sight, since students will surely ask questions for which you must know the answer (although it is sometimes wise not to give it). We therefore list the common area formulas in Figure 5. Note that each formula follows readily from geometric considerations: the parallelogram and right triangle, for example, were mentioned above. The formulas for arbitrary triangles are best understood by cloning the given triangle and forming a parallelogram with the resulting pair of congruent triangles. Finally, a trapezoid can be divided into two triangles whose areas can be calculated separately and added. Sample calculations are given in Figure 6.

But what about an arbitrary polygon? For example, how do we find the area of the nine-sided polygon depicted in Figure 7? After spending a lot of time, perhaps days, calculating the area of squares, rectangles and triangles, most students will break up a complicated polygon like this one into more manageable pieces. In fact, take a moment to do just that: break the nine-



square: $A = s \times s = s^2$

rectangle: $A = b \times h$

parallelogram: $A = b \times h$

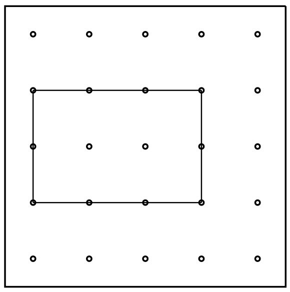
obtuse triangle: $A = \frac{1}{2} \times b \times h$

right triangle: $A = \frac{1}{2} \times b \times h$

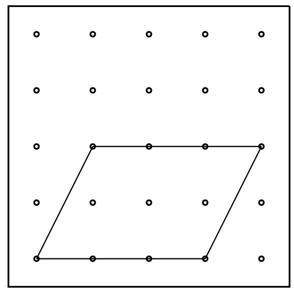
acute triangle: $A = \frac{1}{2} \times b \times h$

trapezoid: $A = \frac{1}{2} \times (a + b) \times h$

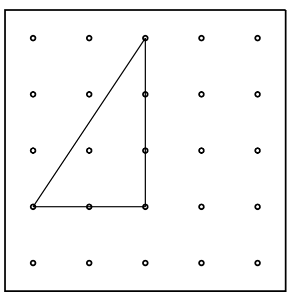
Figure 5: Common area formulas



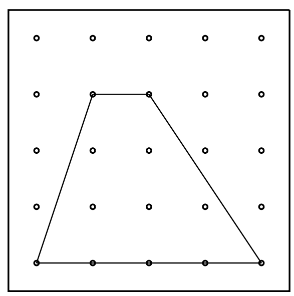
(a) $A = l \times w$
$= 3 \times 2$
$= 6$



(b) $A = b \times h$
$= 3 \times 2$
$= 6$



(c) $A = \frac{1}{2} \times b \times h$
$= \frac{1}{2} \times 2 \times 3$
$= 3$



(d) $A = \frac{1}{2} \times (a + b) \times h$
$= \frac{1}{2} \times (4 + 1) \times 3$
$= \frac{15}{2}$

Figure 6: Sample area calculations

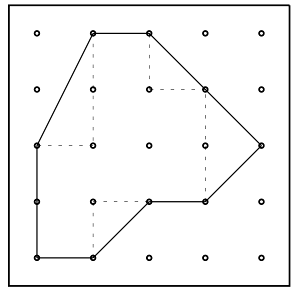


Figure 7: Computing the area of an arbitrary polygon

sided polygon in Figure 7 into squares and triangles, and compute its area. Please, do it now—it's important for what follows.

Okay, good, now let's do it another way. There's a method of calculating the area of any polygon on a geoboard quickly and easily using what's called Pick's theorem. It's almost too good to be true, but Pick's theorem computes the area of an arbitrary polygon simply by counting pegs. Note that the polygon in Figure 7 touches ten pegs and surrounds six pegs. Now if we take the number of boundary pegs (which is 10 in this case) and divide it in half, add the number of interior pegs (which is 6) and subtract one, we get

$$\frac{10}{2} + 6 - 1 = 10$$

which should agree with what you obtained earlier by summing the areas of the polygon's constituent squares and triangles.

Let's check to see that Pick's theorem holds for two of the simple examples in Figure 6. The rectangle in Figure 6a, for instance, has ten boundary pegs and two interior pegs. By Pick's theorem, the area is

$$\frac{10}{2} + 2 - 1 = 6$$

which checks with our earlier calculation via the area formula $A = b \times h$. Similarly, for the parallelogram in Figure 6b, we have

$$\frac{8}{2} + 3 - 1 = 6.$$

We leave it to you to use Pick's theorem to verify the areas of the triangle and trapezoid in Figures 6c and 6d.

Pick's theorem, then, can be stated as follows. Let b be the number of boundary pegs and i be the number of interior pegs of any polygon on the geoboard. Then the area of the polygon is given by

$$\boxed{A = \frac{b}{2} + i - 1}$$

With a little practice, you'll be able to simply look at a polygon and compute its area in your head.

Epilogue

In this introduction, we tried to give you some tools with which to approach this unit on length and area. The Pythagorean theorem, the area formulas in Figure 5, and Pick's area theorem are some of the mathematical tools you'll find useful. Bear in mind, however, that these formulas are not necessarily for student consumption. Indeed, we strongly advise against this. As a general rule, asking students—especially younger kids—to apply a formula such as $A = b \times h$ is not the best way to introduce a concept. It gives them the impression that mathematics is more a matter of memorizing and applying formulas, which it is not. The formula should come later, perhaps much later, after the student has wrestled with concept.

Lesson 1

TBA

Lesson 2

Find all line segments

Objectives

- to develop a sense of length on the geoboard
- to associate attributes of line segments with appropriate terminology (e.g., **horizontal**, **parallel**, **intersecting**, **perpendicular**, **congruent**)
- to develop a working knowledge of order attributes (e.g., **less than**, **greater than**, **between**) with respect to length of line segments
- to experiment with geometric pattern making

Materials

- 5×5 geoboards (the kind that fit together to make a 10×10 geoboard) and rubber bands (various sizes and colors)
- at least five geoboards for the teacher and one for each student
- 5×5 geoboard dot paper (two sheets for each student)
- 10×10 geoboard dot paper (for extended activities)
- overhead projector; transparent geoboard or dot paper (with marking pens)

Warm-up activities

1. Make a horizontal line segment that touches 3 pegs. What is the length of this line segment?
2. Make another line segment that touches 3 pegs, but with different length. Is the length of this new line segment less than or greater than the previous one?
3. Can you find a third line segment that touches 3 pegs, but with length different than the other two. Is the length of this line segment less than or greater than the previous ones?
4. Find the shortest line segment on your geoboard.
5. Find the next shortest line segment.
6. Find the longest line segment on your geoboard.
7. Find the next longest line segment.

Main activity

Find all possible line segments on a 5×5 geoboard.[1]

Homework

Order each of the line segments in the main activity by length.[2]

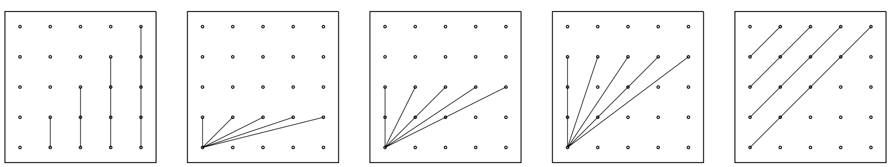
Extended activities

1. On a 5×5 geoboard, there are five line segments with length greater than 4 units. Can you find them?
2. Find three line segments with length less than 3 units. Are there others?

[1] There are a total of 14 such line segments.

[2] In order, the 14 line segments have lengths $\sqrt{1}$, $\sqrt{2}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{8}$, $\sqrt{9}$, $\sqrt{10}$, $\sqrt{13}$, $\sqrt{16}$, $\sqrt{17}$, $\sqrt{18}$, $\sqrt{20}$, $\sqrt{25}$, and $\sqrt{32}$ units.

3. Find two line segments with length between 3 and 4 units.
4. With two rubber bands, make two line segments that touch a total of 9 pegs.
5. Make two parallel line segments that touch a total of 9 pegs.
6. Make two perpendicular line segments that touch a total of 9 pegs.
7. Make two intersecting line segments that touch a total of 9 pegs, but are not perpendicular.
8. Make two congruent line segments that touch a total of 9 pegs. Do these line segments intersect?
9. Make two congruent line segments that touch a total of 8 pegs. Do these line segments intersect?
10. Continue each pattern on a 10×10 geoboard:



11. How many different line segments are there on a 10×10 geoboard?
12. Order each of the line segments in the previous activity by length.

Lesson 3

Find all squares

Objectives

- to begin to develop a sense of area on the geoboard
- to develop a working knowledge of order attributes (e.g., **between**) with respect to area of squares
- to experiment with geometric pattern making

Materials

- 5×5 geoboards (the kind that fit together to make a 10×10 geoboard) and rubber bands (various sizes and colors)
- at least five geoboards for the teacher and one for each student
- 5×5 geoboard dot paper (one sheet for each student)
- 10×10 geoboard dot paper (for extended activities)
- overhead projector; transparent geoboard or dot paper (with marking pens)

Warm-up activities

1. With one rubber band, make a square on your geoboard. What is its area?

2. Find the square on your geoboard with smallest area.

3. Find the square on your geoboard with next smallest area.

4. Find the square on your geoboard with largest area.

Main activity

Find all possible squares on a 5×5 geoboard.[1]

Homework

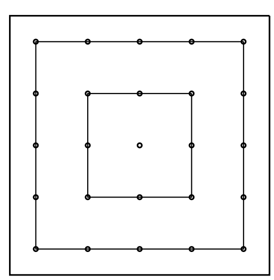
Order each of the squares in the main activity by area.[2]

Extended Activities

1. On a 5×5 geoboard, find the square with next to largest area.

2. Find a square with area between 4 and 9 square units. Can you find another square with area between 4 and 9 square units?

3. What is the area of the shaded region below?



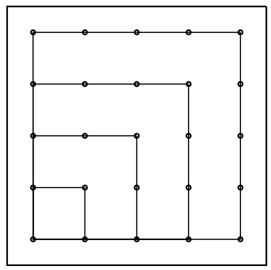
[1] There are a total of 8 such squares.

[2] In order, the 8 squares have area 1, 2, 4, 5, 8, 9, 10, and 16 square units.

4. Find the perimeter of each of the squares in the main activity.

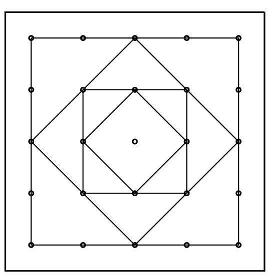
5. On a 10×10 geoboard, make a square using each of the line segments in Lesson 2. Which of these squares do not fit on a 5×5 geoboard?

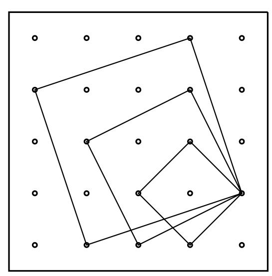
6. Continue the pattern below on a 10×10 geoboard and complete the following table:



length:	1	2	3	4	5	6	7	8	9
area:	1	4	9	16					
number of boundary pegs:	4	8	12	16					
number of interior pegs:	0	1	4	9					

7. Continue these patterns on a 10×10 geoboard:





8. How many possible squares are there on a 10×10 geoboard?

9. Order each of the squares in the previous activity by area.

Lesson 4

Find all rectangles

Objectives

- to continue developing a sense of area on the geoboard
- to realize that all squares are rectangles
- to experiment with geometric pattern making

Materials

- 5×5 geoboards (the kind that fit together to make a 10×10 geoboard) and rubber bands (various sizes and colors)
- at least five geoboards for the teacher and one for each student
- 5×5 geoboard dot paper (two sheets for each student)
- 10×10 geoboard dot paper (for extended activities)
- overhead projector; transparent geoboard or dot paper (with marking pens)

Warm-up activities

1. Make a rectangle on your geoboard. What is its area?

2. How many rectangles are there with base 1 unit long?

3. How many rectangles are there with base 2 units long?

Main activity

Find all possible rectangles on a 5×5 geoboard.[1]

Homework

Order each of the rectangles in the main activity by area.[2]

Extended activities

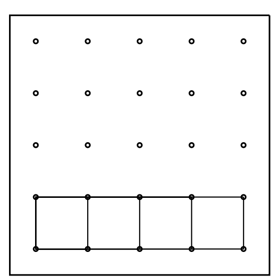
1. Make a rectangle with area 2 square units. Can you find another rectangle with this area?

2. Can you find a rectangle and a square with the same area?

3. There are three rectangles with area 4 square units. Can you find them?

4. Find the perimeter of each of the rectangles in the main activity.

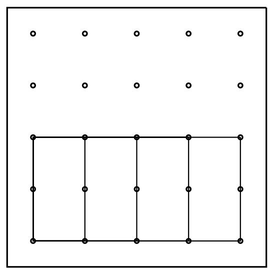
5. Continue this pattern of rectangles on a 10×10 geoboard and complete the table:



[1] There are a total of 16 such rectangles, half of which are squares.
[2] The non-square rectangles have area 2, 3, 4, 4, 6, 6, 8, and 12 square units.

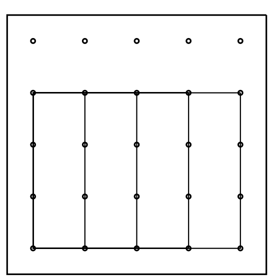
base:	1	2	3	4	5	6	7	8	9
area:	1	2	3	4					
number of boundary pegs:	4	6	8	10					
number of interior pegs:	0	0	0	0					

6. Continue this pattern of rectangles on a 10×10 geoboard and complete the table:



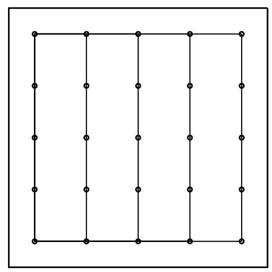
base:	1	2	3	4	5	6	7	8	9
area:	2	4	6	8					
number of boundary pegs:	6	8	10	12					
number of interior pegs:	0	1	2	3					

7. Continue this pattern of rectangles on a 10×10 geoboard and complete the table:



base:	1	2	3	4	5	6	7	8	9
area:	3	6	9	12					
number of boundary pegs:	8	10	12	14					
number of interior pegs:	0	2	4	6					

8. Continue this pattern of rectangles on a 10×10 geoboard and complete the table:



base:	1	2	3	4	5	6	7	8	9
area:	4	8	12	16					
number of boundary pegs:	10	12	14	16					
number of interior pegs:	0	3	6	9					

Lesson 5

Find all parallelograms

Objectives

- to continue developing the student's sense of area, especially the relationship between the area of a parallelogram and the area of the corresponding rectangle
- to realize that all rectangles are parallelograms
- to review some geometric concepts (e.g., **congruent**, **similar**) in the context of parallelograms
- to experiment with geometric pattern making

Materials

- 5×5 geoboards (the kind that fit together to make a 10×10 geoboard) and rubber bands (various sizes and colors)
- at least five geoboards for the teacher and one for each student
- 5×5 geoboard dot paper (two or three sheets for each student)
- 10×10 geoboard dot paper (for extended activities)
- overhead projector; transparent geoboard or dot paper (with marking pens)

Warm-up activities

1. Make a rectangle with base 2 units on your geoboard. Without removing the rectangle, make a parallelogram with the same base.
2. What is the area of the rectangle in the previous activity? What is the area of the parallelogram? How do you know?
3. Find another parallelogram with the same area as the parallelogram in the previous activity.
4. On a 5×5 geoboard, find all parallelograms with a base of 2 units. How many are there?[1]
5. Make a parallelogram with base 1 unit on your geoboard.
6. Find three more parallelograms with base 1 unit having the same area as the parallelogram in the the previous activity. What is the area of each of these parallelograms?
7. Find four more parallelograms with base 1 unit, all having the same area.
8. How many different parallelograms with base 1 unit are there?[2]
9. Find a parallelogram with smallest area. Can you find other parallelograms with this area?
10. Find a parallelogram with next smallest area. Can you find others?
11. Find the parallelogram with largest area.
12. Find the parallelogram with next largest area.

Main activity

Find all possible parallelograms on a 5×5 geoboard.[3]

[1] There are 12 such parallelograms, including four rectangles.

[2] There are 16 such parallelograms, including four rectangles.

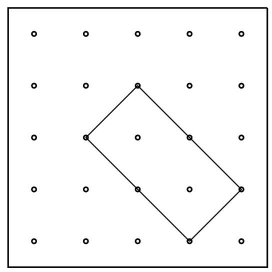
[3] There are 26 non-rectangular parallelograms. Together with the 16 rectangles of Lesson 4, this gives a total of 42 parallelograms.

Homework

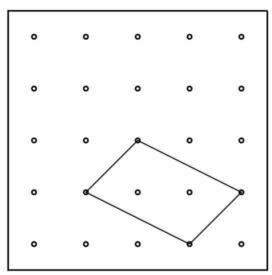
Order each of the parallelograms found in the main activity by area.[4]

Extended activities

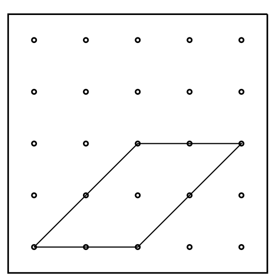
1. Make a parallelogram that has the same area as the following rectangle:



2. Make a parallelogram that is congruent to the one below:



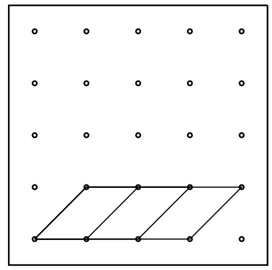
3. Make a parallelogram that is similar to the one below:



[4]Here is a summary of the 26 non-rectangular parallelograms:

Area:	1	2	3	4	5	6	7	8	9	10	11	12
Number:	3	3	3	6	1	4	1	3	1	0	0	1

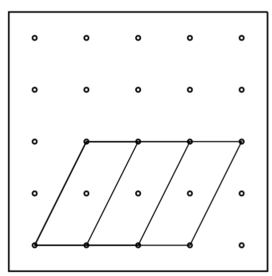
4. Continue this pattern of parallelograms on a 10×10 geoboard and complete the table:



base:	1	2	3	4	5	6	7	8
area:	1	2	3					
number of boundary pegs:	4	6	8					
number of interior pegs:	0	0	0					

Compare your results with extended activity ?? of Lesson 4.

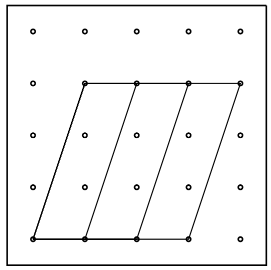
5. Continue this pattern of parallelograms on a 10×10 geoboard and complete the table:



base:	1	2	3	4	5	6	7	8
area:	2	4	6					
number of boundary pegs:	4	6	8					
number of interior pegs:	1	2	3					

Compare your results with extended activity ?? of Lesson 4.

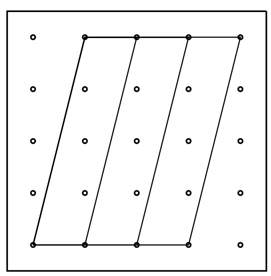
6. Continue this pattern of parallelograms on a 10×10 geoboard and complete the table:



base:	1	2	3	4	5	6	7	8
area:	3	6	9					
number of boundary pegs:	4	6	8					
number of interior pegs:	2	4	6					

Compare your results with extended activity ?? of Lesson 4.

7. Continue this pattern of parallelograms on a 10×10 geoboard and complete the table:



base:	1	2	3	4	5	6	7	8
area:	4	8	12					
number of boundary pegs:	4	6	8					
number of interior pegs:	3	6	9					

Compare your results with extended activity ?? of Lesson 4.

Lesson 6

Find all right triangles

Objectives

- to continue developing the student's sense of area, especially the relationship between the area of a right triangle and the area of the corresponding rectangle
- to associate certain attributes of right triangles with appropriate terminology (e.g., **similar**, **isosceles**)
- to experiment with geometric pattern making

Materials

- 5×5 geoboards (the kind that fit together to make a 10×10 geoboard) and rubber bands (various sizes and colors)
- at least five geoboards for the teacher and one for each student
- 5×5 geoboard dot paper (two sheets for each student)
- 10×10 geoboard dot paper (for extended activities)
- overhead projector; transparent geoboard or dot paper (with marking pens)

Warm-up activities

1. Make a rectangle on your geoboard. What is its area?
2. Make two right triangles from this rectangle. What is the area of each right triangle?
3. Make another right triangle on your geoboard and determine its area.
4. Find the right triangle with smallest area.
5. Find the right triangle with next smallest area.
6. Find the right triangle with largest area.
7. Find the right triangle with next largest area.

Main activity

Find all possible right triangles on a 5×5 geoboard.[1]

Homework

Order each of the right triangles found in the main activity by area.[2]

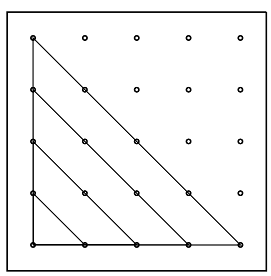
Extended activities

1. Make an isosceles right triangle, namely, a right triangle having two equal sides.
2. Make another isosceles right triangle, but with different area. Is this triangle similar to the first one?
3. Find another pair of isosceles right triangles (with different area).
4. Find four similar right triangles, all with different area.

[1] There are a total of 17 right triangles.

[2] In order, the 17 right triangles have area $\frac{1}{2}$, 1, 1, $\frac{3}{2}$, 2, 2, 2, $\frac{5}{2}$, 3, 3, 4, 4, $\frac{9}{2}$, 5, 5, 6, and 8 square units.

5. Make a right triangle that is not isosceles.
6. Make two right triangles which are similar, but not isosceles.
7. Make a right triangle with area 1 square unit. Can you find another right triangle with area 1 square unit?
8. Find four right triangles with area less than 2 square units.
9. There are three right triangles with area 2 square units. Can you find them?
10. Find a right triangle with area $2\frac{1}{2}$ square units.
11. Find a right triangle with area 3 square units. Can you find another right triangle with area 3 square units?
12. How many right triangles can you find with area 4 square units?
13. Find a right triangle with area $4\frac{1}{2}$ square units.
14. Find two right triangles with area 5 square units.
15. Continue this pattern of triangles on a 10×10 geoboard and complete the table:



base:	1	2	3	4	5	6	7	8	9
area:	$\frac{1}{2}$	2	$\frac{9}{2}$	8					
number of boundary pegs:	3	6	9	12					
number of interior pegs:	0	0	1	3					

Bibliography

[1] John Bradford (1987). *Geoboard Teacher's Manual*. Ft. Collins, CO: Scott Resources, Inc. (Bradford's manual is an excellent source of geoboard-related ideas.) [QA 445 B799]

[2] Marilyn Burns (1992). *About Teaching Mathematics: A K–8 Resource*. White Plains, NY: Cuisenaire Company of America. (Includes many interesting activities on pp. 95–99, including a discovery lesson on Pick's theorem.) [QA 135.5 B87]

[3] Dorothy Geddes et al. (1992). *Geometry in the Middle Grades*. Reston, VA: National Council of Teachers of Mathematics. (Activity 5D on p. 37 is entitled "Patterns on a Geoboard"; Activity 5E, "Pythagorean Theorem", also uses the geoboard.) [QA 461 G43]

[4] Ryan McElduff and Michael Oberdorf (1992). *Try It! Geoboards*. White Plains, NY: Cuisenaire Company of America. (Includes 24 cards of challenging problems, which are really quite easy if you know Pick's theorem.) [QA 445 M14]

[5] Mark A. Spikell (1993). *Teaching Mathematics with Manipulatives: A Resource of Activities for the K–12 Teacher*. Boston: Allyn and Bacon. (Chapter 2 is entitled "Motivating the Pythagorean Theorem with Geoboards".) [QA 18 S65]

[6] John V. Trivett (1983). *Introducing Geoboards*. New Rochelle, NY: Cuisenaire Company of America. (Consists of 56 cards of basic introductory material) [QA 443 T841]

[7] Rosamond Welchman-Tischler (1992). *The Mathematical Toolbox*. White Plains, NY: Cuisenaire Company of America. (Challenge Set #3 contains many good geoboard-related ideas.) [QA 139 W44]