Some results on tangential triangle

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Let AtBtCt be the tangential triangle of triangle ABC, as shown in Fig. 1.

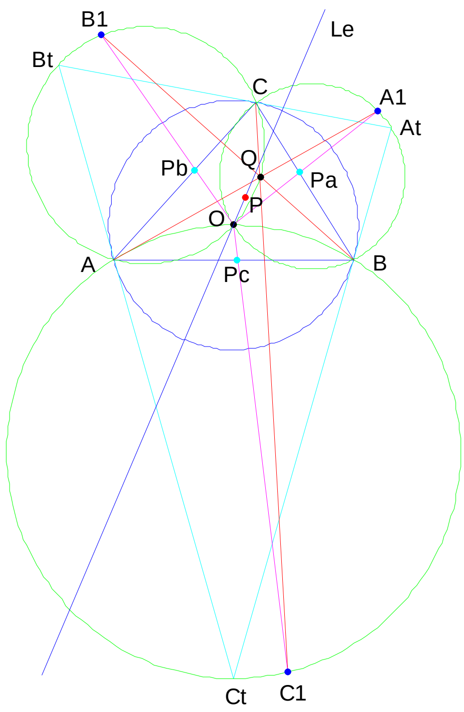

Fig. 1

Let P = (1-t)*O + t*G be an arbitrary point on Euler line Le of triangle ABC, where O and G are respectively the circumcenter and centroid of triangle ABC. Let Pa, Pb, Pc be the traces of P on the respective sides of triangle ABC. Let A1, B1, C1 be the intersection points of lines OPa, OPb, OPc with the circumcircles of triangles AtBC, ABtC, ABCt respectively. Then lines AA1, BB1, CC1 are concurrent at the point Q with barycentric coordinates (u : v : w), where

$$u = a^2 \cdot ((1-t)\cdot 3\cdot a^2 \cdot (b^2 + c^2 - a^2) + t\cdot 4\cdot S^2)\cdot ((1-t)\cdot 3\cdot b^2 \cdot c^2 + t\cdot 4\cdot S^2)\ , \qquad (1)$$

where S is twice the area of triangle ABC.

From (1) follows that O is a fixed point, and for t = 1, that means P = G, we obtain Q = X(6), the Symmedian point.
For t = 3/2, that means P = X(5) , the Nine-point center of triangle ABC, we obtain Q = X(143), the Nine-point center of orthic triangle of triangle ABC.

Let Circ(At), Circ(Bt), Circ(Ct) be the circumcircles of triangles AtBC, ABtC, ABCt respectively, see Fig. 2.

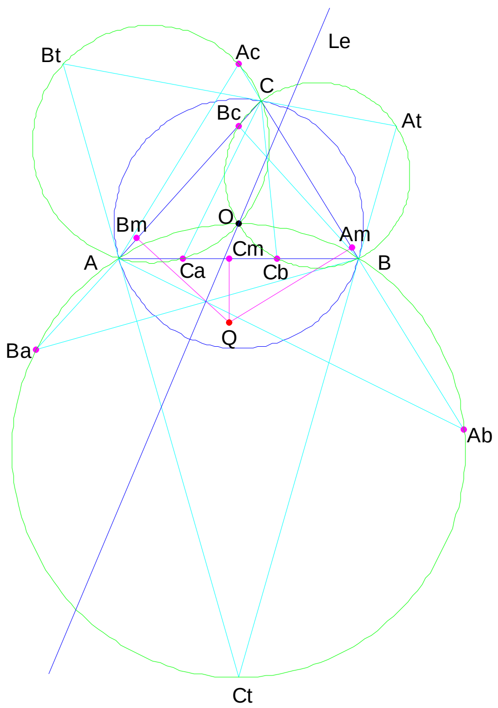

Fig. 2

Let Ac, Ab be the intersection points of the line BC with Circ(Bt), Circ(Ct) respectively. Points Ac, Ab are also the intersection points of line BC with lines CtO, BtO respectively. The barycentric coordinates of Ac, Ab are

$$\begin{aligned} Ac &= (\; 0 \; : a^2 - b^2 : \; c^2 \;) \\ Ab &= (\; 0 \; : \; b^2 \; : a^2 - c^2) \end{aligned} \quad . \tag{2}$$

Let Am be the middle point of segment AcAb. Similarly we obtain Bm, Cm as middle points of segments BaBc, CbCa respectively. The lines through Am, Bm, Cm and perpendicular to the sides BC, CA, AB respectively, are concurrent at the point Q with barycentric coordinates (u : v : w), where

$$u = a^2 \cdot ((c^2 - a^2) \cdot b^2 \cdot S_b^2 + (b^2 - a^2) \cdot c^2 \cdot S_c^2 + a^4 \cdot S_a^2) \ , \qquad (3)$$

where $S_a = b^2 + c^2 - a^2$, $S_b = c^2 + a^2 - b^2$, $S_c = a^2 + b^2 - c^2$.

The intersection points of lines AAb and BBa , BBc and CCb, CCa and AAc are collinear and lie on the Euler line Le of triangle ABC. The circumcenter of tangential triangle AtBtCt also lies on Le.

Triangles AtBtCt, AAbAc, BBcBa, CCaCb are similar and their respective incircles are concentric with center O, the circumcenter of triangle ABC.