Some results on side-bisector reflected triangle

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Let ABC be an arbitrary triangle. Let A', B', C' be the reflections of vertices A, B, C on the bisectors of sides BC, CA, AB respectively. Points A', B', C' form a new triangle $A'B'C'$ and have barycentric coordinates as follows

$$\begin{aligned} A' &= (\ a^2 : c^2 - b^2 : b^2 - c^2) \\ B' &= (c^2 - a^2 : \ b^2 \ : a^2 - c^2) \\ C' &= (b^2 - a^2 : a^2 - b^2 : \ c^2 \ ) \end{aligned} \quad . \tag{1}$$

The lines through vertices A', B', C' and perpendicular respectively to the sides BC, CA, AB of triangle ABC, are concurrent at the *de Longchamps point* X(20) of triangle ABC, see Fig.1.

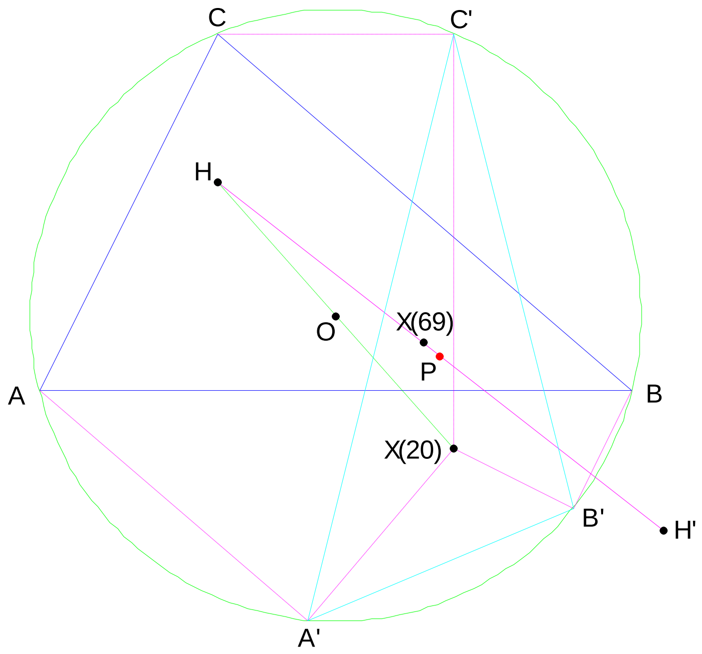

Fig. 1

Let A'_p, B'_p, C'_p be the orthogonal projections of vertices A', B', C' respectively to the sides BC, CA , AB of triangle ABC. Then lines AA'_p, BB'_p, CC'_p are concurrent at the point X(69) of triangle ABC.

The *Simson lines* of vertices A', B', C' with respect to triangle ABC, are concurrent at the point P = (u : v : w), where

$$u = a^2 \cdot S_a \cdot [\, 2 \cdot a^2 \cdot b^2 \cdot c^2 + b^2 \cdot S_b \cdot (b^2 - a^2) + c^2 \cdot S_c \cdot (c^2 - a^2)] \,, \qquad (2)$$

where $S_a = b^2 + c^2 - a^2$, $S_b = c^2 + a^2 - b^2$, $S_c = a^2 + b^2 - c^2$.

Let H, H' be the orthocenters of triangles ABC, $A'B'C'$ respectively. Then points H, X(69), P, H' are collinear, and $\overline{HP} = \overline{PH'}$, see Fig 1.

Now we can repeat the construction of point P for the triangle $A'B'C'$ as starting triangle, and so on, after 10000 iterations we obtain a distribution of points P as shown in Fig. 2, the red points.

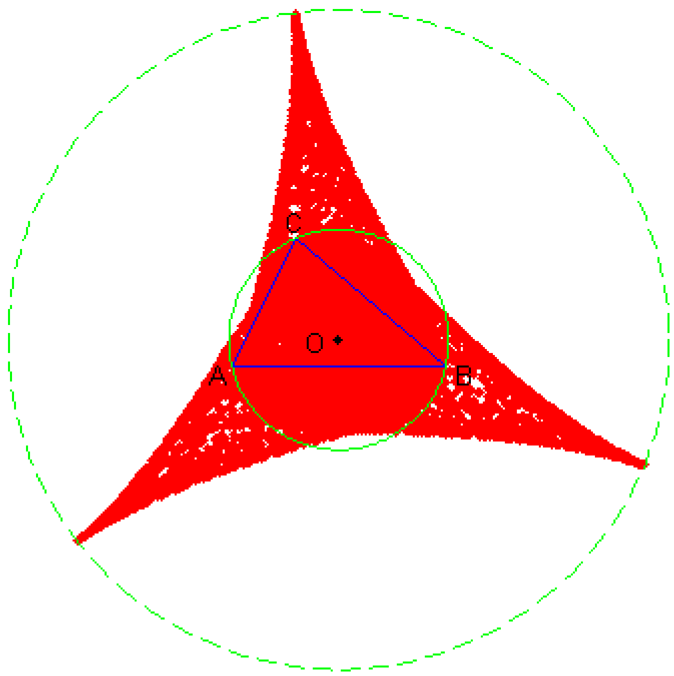

Fig. 2

The red P points lie inside the circle with center at the orthocenter O of triangle ABC, and radius 3*R, where R is the circumradius of triangle ABC.

Next we find the envelope of the red P points. We start from the equations of the *deltoid*[1]

$$\begin{aligned} x &= 3R[\,\frac{2}{3}cos(t) + \frac{1}{3}cos(2t)] \\ y &= 3R[\,\frac{2}{3}sin(t) - \frac{1}{3}sin(2t)] \end{aligned} \,. \qquad (3)$$

[1] Deltoid: http://mathworld.wolfram.com/Deltoid.html

The slope of deltoid (3) for an arbitrary angle $t = \varphi$, is

$$\frac{dy}{dx} = \frac{dy/d\varphi}{dx/d\varphi} = -\tan\frac{\varphi}{2} \quad . \tag{4}$$

Let $D_1 = [x(\varphi), y(\varphi)]$ and $D_2 = [x(\varphi+\pi), y(\varphi+\pi)]$ be two points in deltoid (3). The line D_1D_2 has the slope $s = \tan(\varphi)$, or $s = -\tan(-2\varphi/2)$. This means that the line D_1D_2 has the same slope as the tangent of deltoid (3) at the point $D_3 = [x(-2\varphi), y(-2\varphi)]$. The points D_1, D_2, D_3 are collinear, because

$$\begin{vmatrix} x(\varphi) & y(\varphi) & 1 \\ x(\varphi+\pi) & y(\varphi+\pi) & 1 \\ x(-2\varphi) & y(-2\varphi) & 1 \end{vmatrix} = 0 \quad . \tag{5}$$

Now we define the point $D_4 = (D_1 + D_3)/2$, so we have

$$\begin{aligned} x &= \frac{3R}{2}[\frac{2}{3}\cos(\varphi) + \cos(2\varphi) + \frac{1}{3}\cos(4\varphi)] \\ y &= \frac{3R}{2}[\frac{2}{3}\sin(\varphi) - \sin(2\varphi) + \frac{1}{3}\sin(4\varphi)] \end{aligned} \quad . \tag{6}$$

The curve defined by equations (6) we call the *bell deltoid*, see Fig. 3.

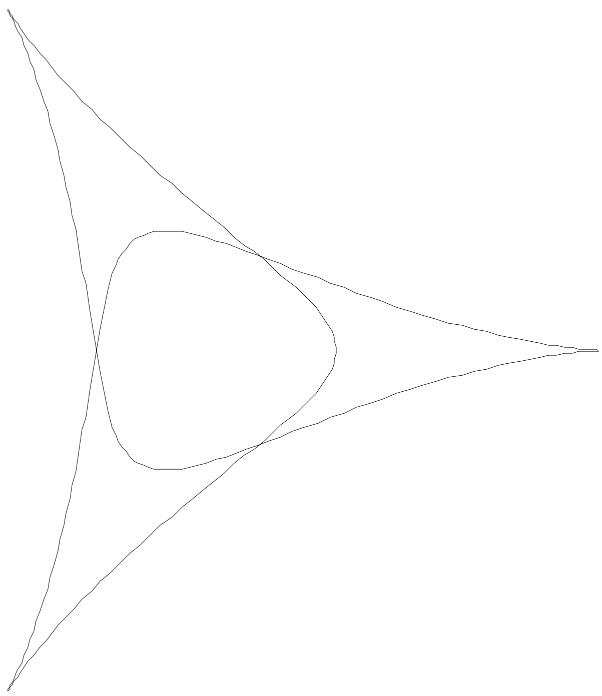

Fig. 3

Finally the envelope of the red points P is a *bell deltoid* with orientation opposite to the *first Morley triangle*[2], and it is shown in Fig. 4.

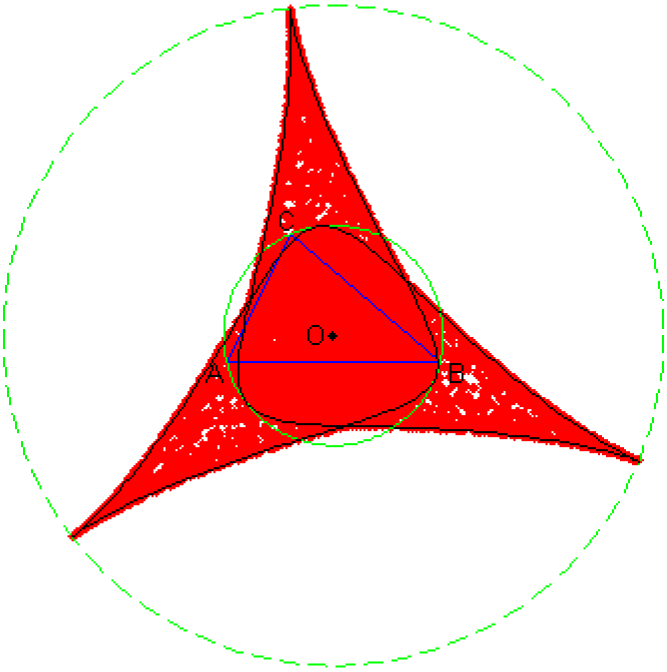

Fig. 4

[2] Simson Line: http://mathworld.wolfram.com/SimsonLine.html