Proving Irrationality by the Rational Roots Theorem

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RATIONAL ROOT THEOREM: Let $P(X)$ be a polynomial with integer coefficients, say

$$P(X) = A_nX^n + A_{n-1}X^{n-1} + A_{n-2}X^{n-2} + \cdots + A_2X^2 + A_1X + A_0.$$

If $\frac{p}{q}$ is a quotient of integers in lowest terms and $P\left(\frac{p}{q}\right) = 0$, then p is a divisor of A_0 and q is a divisor of A_n.

PROOF: Assume that $\frac{p}{q}$ is a quotient of integers in lowest terms and that $P\left(\frac{p}{q}\right) = 0$.[1] Then

$$A_n\left(\frac{p}{q}\right)^n + A_{n-1}\left(\frac{p}{q}\right)^{n-1} + A_{n-2}\left(\frac{p}{q}\right)^{n-2} + \cdots + A_2\left(\frac{p}{q}\right)^2 + A_1\left(\frac{p}{q}\right) + A_0 = 0.$$

Rewriting and then multiplying both sides by q^n (to clear out the denominators) gives

$$\frac{A_np^n}{q^n} + \frac{A_{n-1}p^{n-1}}{q^{n-1}} + \frac{A_{n-2}p^{n-2}}{q^{n-2}} + \cdots + \frac{A_2p^2}{q^2} + \frac{A_1p}{q} + A_0 = 0$$

$$A_np^n + A_{n-1}p^{n-1}q + A_{n-2}p^{n-2}q^2 + \cdots + A_2p^2q^{n-2} + A_1pq^{n-1} + A_0q^n = 0.$$

(p is a divisor of A_0) Isolate the term containing A_0:

$$\begin{aligned} A_0q^n &= -A_np^n - A_{n-1}p^{n-1}q - A_{n-2}p^{n-2}q^2 - \cdots - A_2p^2q^{n-2} - A_1pq^{n-1} \\ &= p \cdot \left(-A_np^{n-1} - A_{n-1}p^{n-2}q - A_{n-2}p^{n-3}q^2 - \cdots - A_2pq^{n-2} - A_1q^{n-1}\right). \end{aligned}$$

This shows that p is a factor of A_0q^n. Now consider the prime factors of p.[2] Because $\frac{p}{q}$ is in lowest terms, none of the prime factors of p will appear among the prime factors of q. Therefore, none of the prime factors of p will appear among the prime factors of q^n, since the prime factors of q^n are the same as the prime factors of q (you just have n times as many of them in q^n as you do in q). This means that all the prime factors of p show up in A_0. Hence, p divides A_0.

[1] There is a technical problem if $\frac{p}{q} = 0$. This is because no matter what "lowest terms" might mean in this case, we must have $p = 0$. But now p can't be a divisor of A_0 (or of any number, for that matter), since division by 0 is undefined. However, it is obvious when 0 is a root—when $A_0 = 0$. Therefore, we will assume that $\frac{p}{q} \neq 0$. Note this implies $p \neq 0$. (We will need this later.)

[2] If p has no prime factors, then $p = 1$ or $p = -1$ (remember, $p \neq 0$ by our previous footnote). In either case, p is a divisor of A_0.

(q is a divisor of A_n) Isolate the term containing A_n:

$$\begin{aligned} A_n p^n &= -A_{n-1}p^{n-1}q - A_{n-2}p^{n-2}q^2 - \cdots - A_2p^2q^{n-2} - A_1pq^{n-1} - A_0q^n \\ &= q \cdot \left(-A_{n-1}p^{n-1} - A_{n-2}p^{n-2}q - \cdots - A_2p^2q^{n-3} - A_1pq^{n-2} - A_0q^{n-1}\right). \end{aligned}$$

This shows that q is a factor of A_np^n. Now consider the prime factors of q.[3] Because $\frac{p}{q}$ is in lowest terms, none of the prime factors of q will appear among the prime factors of p. Therefore, none of the prime factors of q will appear in p^n, since the prime factors of p^n are the same as the prime factors of p (you just have n times as many of them in p^n as you do in p). This means that all the prime factors of q show up in A_n. Hence, q divides A_n.

EXAMPLE 1: Show that $\sqrt{3}$ is irrational.

$\sqrt{3}$ is a root of the polynomial $x^2 - 3$. By the above theorem, the only candidates for rational root(s) of $x^2 - 3$ are

$$\frac{p}{q} = \pm 1,\ \pm 3.$$

Here are two methods you can use to conclude that $\sqrt{3}$ is irrational. **(1)** None of the rational candidates are roots of $x^2 - 3$ (just plug in each of these four numbers and see that you don't get 0) , so this polynomial doesn't have *any* rational roots. Since $\sqrt{3}$ is a root of $x^2 - 3$, it follows that $\sqrt{3}$ cannot be rational. **(2)** $\sqrt{3}$ is approximately 1.732, which is clearly different from any of the rational root candidates. Therefore, $\sqrt{3}$ cannot be rational.

REMARK: Method (1) is the quickest when it is relatively easy to see that none of the candidates are roots. However, if you have a lot of possibilities for $\frac{p}{q}$, or if some of the candidates *happen to be true roots* (a polynomial can have both rational and irrational roots—consider $(x-2)\cdot\left(x^2-3\right)$), then you'll need to use method (2).

EXAMPLE 2: Show that $2 + \sqrt{3}$ is irrational.

In this case, you might not immediately know of a polynomial having this number as a root. When this happens, call the number x:

$$x = 2 + \sqrt{3}\,.$$

Isolate the radical, square both sides, and then get a 0 on one side:

$$x - 2 = \sqrt{3}$$

$$(x-2)^2 = 3$$

$$(x-2)^2 - 3 = 0.$$

This shows that $(x-2)^2 - 3$ (which equals $x^2 - 4x + 1$) is a polynomial with $2 + \sqrt{3}$ as a root. The rest is now easy—easier than example 1, in fact.

EXAMPLE 3: Show that $\sqrt{2} + \sqrt{3}$ is irrational.

Let x be this number. What we want to do is reverse the steps of solving an equation, beginning with its solution and ending with the equation. How do we know when we are done? When you have a polynomial equation with integer coefficients equal to 0. A helpful strategy is to see if you can get just one radical. Once you get to this point, isolate

[3] If q has no prime factors, then $q = 1$ or $q = -1$ ($q \neq 0$ since q is the denominator of a fraction). In either case, q is a divisor of A_n.

that radical and then square (or cube, or whatever, depending on what the radical is) both sides.

$$\begin{aligned}
x &= \sqrt{2}+\sqrt{3}\\
x^2 &= \left(\sqrt{2}+\sqrt{3}\right)^2\\
x^2 &= 2+2\sqrt{6}+3\\
x^2-5 &= 2\sqrt{6}\\
\left(x^2-5\right)^2 &= \left(2\sqrt{6}\right)^2\\
\left(x^2-5\right)^2 - 24 &= 0.
\end{aligned}$$

Therefore, a polynomial having $\sqrt{2}+\sqrt{3}$ as a root is $\left(x^2-5\right)^2-24 = x^4-10x^2+1$. It is now easy to finish the job.

EXAMPLE 4: Show that $\sqrt[3]{2}+\sqrt{3}$ is irrational.

It is probably best to isolate the cube root and then cube both sides.[4]

$$\begin{aligned}
x &= \sqrt[3]{2}+\sqrt{3}\\
x-\sqrt{3} &= \sqrt[3]{2}\\
\left(x-\sqrt{3}\right)^3 &= \left(\sqrt[3]{2}\right)^3\\
x^3-3x^2\sqrt{3}+3x\sqrt{9}-\sqrt{27} &= 2\\
x^3-3x^2\sqrt{3}+9x-3\sqrt{3} &= 2\\
x^3+9x-2 &= \left(3\sqrt{3}\right)\cdot\left(x^2+1\right)\\
\left(x^3+9x-2\right)^2 &= 9\cdot 3\cdot\left(x^2+1\right)^2\\
\left(x^3+9x-2\right)^2-27\left(x^2+1\right)^2 &= 0.
\end{aligned}$$

Now if you want to multiply this out and combine like terms, feel free to do so. But notice that we only need the leading and trailing coefficients in order to construct rational root candidates. These are easy to find:

$$\left(x^6+\cdots+4\right)-27\left(x^4+\cdots+1\right) \;=\; x^6+\cdots+(-23)\,.$$

Therefore, p is a divisor of -23 and q is a divisor of 1. This implies that the rational root candidates are

$$\frac{p}{q} \;=\; \pm 1,\ \pm 23.$$

Rather than plugging these into the polynomial above (yuck!), just notice that none of these can possibly equal $\sqrt[3]{2}+\sqrt{3}$, since this number is obviously not an integer. (And if this had not been obvious, as might be the case in another situation, use a calculator to show that this number—approximately 2.99197—is different from each of the rational root candidates.)

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[4] As a challenge, see if you can get anywhere by first isolating the square root and then squaring. You'll get

$$x^2-\left(2\sqrt[3]{2}\right)\cdot x+\sqrt[3]{4}=3.$$

(The challenge is to somehow get rid of these cube roots.)