Evaluating Limits and Giving Reasons

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SOME BASIC LIMITS

1. $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$	2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$	3. $\lim\limits_{n\to\infty} a^{\frac{1}{n}} = 1 \ (a > 0)$
4. $\lim\limits_{n\to\infty} a^n = 0 \ (-1 < a < 1)$	5. $\lim\limits_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$	6. $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$

1. $\frac{n + (-1)^n}{n}$ CONVERGES to 1. Here are two ways to justify this. **(1)** $\frac{n+(-1)^n}{n} = 1 + \frac{(-1)^n}{n} \longrightarrow 1 + 0 = 1$. **(2)** $\frac{n-1}{n} \le \frac{n+(-1)^n}{n} \le \frac{n+1}{n}$, and the ends each approach 1 [i.e. $\frac{n-1}{n} = \frac{1-\frac{1}{n}}{1} \longrightarrow \frac{1-0}{1} = 1$ (you can also use L'Hôpital's rule), and similarly for $\frac{n+1}{n}$], so the squeeze theorem for sequences says that the middle expression also approaches 1.

2. $\frac{\sin n}{n}$ CONVERGES to 0. Use the squeeze theorem for sequences: $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$, and the ends each approach 0, so the middle expression also approaches 0.

3. $\frac{\sin^2 n}{n}$ CONVERGES to 0. Use the squeeze theorem for sequences: $0 \le \frac{\sin^2 n}{n} \le \frac{1}{n}$, and the ends each approach 0 (the left end represents the constant sequence 0, 0, 0, ...), so the middle expression also approaches 0.

4. $\sqrt{\frac{2n}{n+1}}$ CONVERGES to $\sqrt{2}$. Use continuity of the function $\sqrt{x}$ to take the limit inside the radical: $\lim\limits_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim\limits_{n\to\infty} \frac{2n}{n+1}} = \sqrt{2}$. The limit under the radical should be obvious to anyone in this class, but you should also be able to further justify why it equals 2 if pressed. (See the comment in #1 above).

5. $\sin\left(\frac{\pi}{2} + \frac{1}{n}\right)$ CONVERGES to 1. Use continuity of the function $\sin x$ to take the limit under the sine function: $\lim\limits_{n\to\infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) = \sin\left(\lim\limits_{n\to\infty}\left(\frac{\pi}{2} + \frac{1}{n}\right)\right) = \sin\left(\frac{\pi}{2} + 0\right) = 1$.

6. $\ln(n) - \ln(n+1)$ CONVERGES to 0. Rewrite and then use continuity of the logarithm function: $\lim\limits_{n\to\infty} [\ln(n) - \ln(n+1)] = \lim\limits_{n\to\infty}\left[\ln\left(\frac{n}{n+1}\right)\right] = \ln\left[\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)\right] = \ln(1) = 1$. DO NOT TRY TO JUSTIFY THIS BY SAYING $\ln(n) - \ln(n+1)$ is $\infty - \infty = 0$!! The expression $\infty - \infty$ is indeterminate and can correspond to any number (including $-\infty$ or $+\infty$), or even fail to have a value [$(n + \sin n) - (n)$ is an $\infty - \infty$ form that has no limit].

7. $\frac{\ln n}{\ln 2n}$ CONVERGES to 1. You can either use L'Hôpital's rule, getting $\lim\limits_{n\to\infty} \frac{\ln n}{\ln 2n} = \lim\limits_{n\to\infty} \frac{n^{-1}}{n^{-1}} = 1$, or you can do it directly this way: $\frac{\ln n}{\ln 2n} = \frac{\ln n}{\ln 2 + \ln n} =$ [now divide the numerator and denominator by the term that approaches ∞ the fastest, namely $\ln n$] $\frac{1}{\frac{\ln 2}{\ln n} + 1} \longrightarrow \frac{1}{0+1} = 1$.

8. $\sqrt[n]{10n}$ CONVERGES to 1. Make use of the useful limits in the table above, $\sqrt[n]{a} \to 1$ and $\sqrt[n]{n} \to 1$, along with other properties of limits: $\sqrt[n]{10n} = \left(\sqrt[n]{10}\right)\left(\sqrt[n]{n}\right) \longrightarrow 1 \cdot 1 = 1$.

9. $\sqrt[n]{n^2}$ CONVERGES to 1. This is similar to #8. $\sqrt[n]{n^2} = \left(\sqrt[n]{n}\right)^2 \longrightarrow 1^2 = 1$, where I used the continuity of x^2 to take the limit inside the squaring function. (You could also write this as $\left(\sqrt[n]{n}\right) \cdot \left(\sqrt[n]{n}\right)$, and then take the limit of each factor.)

10. $\left(\frac{3}{n}\right)^{\frac{1}{n}}$ CONVERGES to 1. Again, this is similar to #8: $\left(\frac{3}{n}\right)^{\frac{1}{n}} = \frac{\sqrt[n]{3}}{\sqrt[n]{n}} \longrightarrow \frac{1}{1} = 1$.

11. $\left(\frac{1}{n}\right)^{\frac{1}{\ln n}}$ CONVERGES to e^{-1}. This is an indeterminate 0^0 form that doesn't seem to have an obvious re–writing into something in the table above, so we use the logarithm method. Let $y = \left(\frac{1}{n}\right)^{\frac{1}{\ln n}}$. Then $\ln y = \frac{1}{\ln n} \cdot \ln\left(\frac{1}{n}\right) = \frac{-\ln n}{\ln n} = -1$. Hence, $y = e^{-1}$. Therefore, this weird sequence is just the CONSTANT sequence e^{-1}, e^{-1}, e^{-1}, ... in disguise!

12. $\frac{n!}{2^n \cdot 3^n}$ DIVERGES (to ∞): $\frac{n!}{2^n \cdot 3^n} = \frac{n!}{6^n} \longrightarrow \infty$ by #6 in the table above. Note that the next two problems (#13, 14) are variations of this type.

13. $\sqrt[n]{\frac{3^n}{n}}$ CONVERGES to 3. $\sqrt[n]{\frac{3^n}{n}} = \frac{\sqrt[n]{3^n}}{\sqrt[n]{n}} = \frac{3}{\sqrt[n]{n}} \longrightarrow \frac{3}{1} = 3$.

14. $\frac{(-4)^n}{n!}$ CONVERGES to 0. Use the squeeze theorem for sequences along with one of the "common limits" in table 9.1: $-\frac{4^n}{n!} \le \frac{(-4)^n}{n!} \le \frac{4^n}{n!}$, and the ends each approach 0 (#6 in table above), so the middle expression also approaches 0.

SOME GENERAL REMARKS

- Please don't try to develop an "arithmetic with ∞". Expressions such as $\frac{\infty^2-4}{3\cdot\infty}$ have no place in this class. For us, ∞ represents a PROCESS, not a number. I've given you plenty of mathematically precise techniques for you to justify your work.

- Know which forms are indeterminate and which are not: 1^∞, ∞^0, 0^0, $\infty - \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ are indeterminate forms and ∞^∞, $\infty + \infty$, 0^∞, $\infty \cdot \infty$ are NOT indeterminate forms.

- Use mathematical notation properly. The symbol $=$ is used between mathematical expressions and the symbol $\Rightarrow$ is used between statements. They are not interchangeable. Also, use $=$ only when the expressions really are equal. DON'T WRITE LIES ... $\frac{x^2}{\ln x}$ is not equal to $\frac{2x}{x^{-1}}$ and $\frac{3^n}{n!}$ is not equal to 0!

- Don't write down a bunch of gibberish that serves no purpose. If you don't know how to work the problem right off the bat, use scratch paper. When you're justifying something, have a plan and follow it. Don't just randomly cite facts. Only include those facts that justify, or help explain to the reader, what you're doing. For example, to evaluate $\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}}$, I see no purpose what–so–ever in rewriting $\sqrt{\frac{2n}{n+1}}$ as $\frac{\sqrt{2}\sqrt{n}}{\sqrt{n+1}}$. The way it was originally written makes it an automatic "take the limit under the radical sign" and then evaluate the trivial $\frac{\infty}{\infty}$ form $\lim_{n\to\infty}\left(\frac{2n}{n+1}\right)$. By writing things as $\frac{\sqrt{2}\sqrt{n}}{\sqrt{n+1}}$, you've turned it into a *more difficult* $\frac{\infty}{\infty}$ form.