A Model for Reasoning with Recursion and Mathematical Induction in School Mathematics

Introduction

Recursion and mathematical induction should be explicitly studied in school mathematics classrooms and curricula. As such, reasoning with recursion, recursive thinking and mathematical induction can, and should be, a natural process across grade levels and across multiple strands in school mathematics. For example, strands and suggestions of specific topics within those strands that lend themselves to this type of reasoning include the following:

Geometry: figurate numbers and fractals

Algebra: handshake problems, Tower of Hanoi problems, and proofs by mathematical induction

Discrete mathematics: counting problems and combinatorics

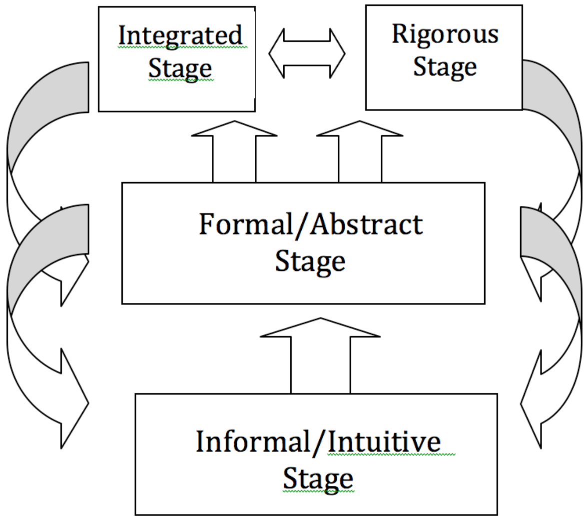
Statistics: demographic problems

A model for reasoning about the ways recursion, recursive thinking, and mathematical induction might be included across the grade levels in school mathematics is proposed in this brief. This model can be helpful in instructional planning as a frame for developing specific lessons.

Introducing a model for developing recursive thinking across grade levels

Each time learners are introduced to a new situation or a new context, they typically interpret or "make sense" of the situation from an informal/intuitive level. This informal/intuitive stage is key to students developing more formal and abstract notions of recursion and mathematical induction. Figure 1 depicts a proposed model for this progression.

Figure 1. Model for reasoning with recursion and mathematical induction



The stages of the model in Figure 1 are described below, followed by examples illustrating aspects of the stages.

Informal/Intuitive stage

Students begin to grasp the idea of recursion by participating in games, contexts, or situations where recursion helps explain what is happening. Recursive notions at this stage may be expressed verbally without algebraic notation. Experiences in this stage provide seeds of ideas for later stages.

Illustrative Example 1. *Run to N* (see example below) is a game where recursive thinking can be used to discover a winning strategy. The first step in generalization is the players' realization that a winner can be declared when a certain number smaller than 20 is reached. When the players realize that more than one number smaller than 20 allows a winner to be declared and when they begin to see a relationship among these numbers and 20, they are at the informal/intuitive stage of recursive reasoning**.** Players who can describe this relationship among numbers numerically or verbally exhibit that they are in this stage.

Once students begin to record verbal ideas and possibly begin building multiple representations with technology to think about their ideas, they are moving from the informal/intuitive stage to the formal/abstract stage. The learner's reasoning is a primary tool for moving between these stages. Group discussions or teacher prompts may facilitate the development of the reasoning allowing a learner to move between the stages. For example in *Run to N*, students might use technology to explore whether or not relationships found work with all constraints in the game. When the learners generalize a relationship to fully describe a determination of the game's winner, they are moving to the formal/abstract stage.

Formal/abstract stage

At this stage, learners represent the idea of recursion explicitly with algebraic notation. For example, they use mathematical expressions to represent the relations among the $(n - 1)$st, the nth, and the $(n + 1)$st terms. It is at this stage that learners may find a general term to mathematize the model, by reasoning about the relationships among the terms. And it is at this stage that learners may extend reasoning patterns into arguments that these relationships hold in general within the given context of the problems.

As learners begin recognizing that mathematization in one strand of mathematics may be similar to the mathematization in another strand of mathematics and that relationships built in one context may be useful in a different context, they are moving to the integrated stage. To help them move to this more generalized stage, records of the verbal ideas and building multiple representations with the aid of technology might be introduced to help visualize patterns among different contexts. Again a primary tool for moving between the stages is the learner's own reasoning. For teachers, it is important to recognize when learners are beginning to move between the stages and to facilitate this movement.

Integrated Stage

In the integrated stage, learners connect the idea of recursion and recursive thinking among mathematical strands. For example, learners at this stage would recognize that the solution to the handshake problem (See Illustrative Example 2) is a problem that may be solved with combinatorics notions and that the same type of thinking may be used to solve other problems of this type. Learners moving to the rigorous recursion stage exhibit a careful consideration of artifacts of earlier stages and begin to consider how these artifacts may be built into a generalized argument or proof that the process used will work in all similar problems and situations. The consideration of other similar problems presented by a teacher may help learners clarify their generalized thinking at this juncture. Organizing and presenting such similar problems requires that teachers have a deep understanding of the mathematics being studied at the time as well as mathematics across strands.

Rigorous recursion stage

Learners in this stage are able to prove mathematical propositions using mathematical induction. It is in this stage that they link the recursive thinking used earlier to develop proofs, and a natural way to produce such generalized proofs is through the use of mathematical induction.

Final Remarks

The learning model, together with the two illustrative examples, suggests that experiences with reasoning through recursion, induction, and mathematical induction should be included throughout the school mathematics curriculum. The illustrative examples serve to explicate processes surrounding reasoning through recursion and induction. The model with its examples may be used as (a) an evaluation tool for helping determine whether curriculum documents include and foster reasoning through recursion and induction; (b) a framework for the development of instructional lessons; and (c) a model for teacher preparation where both inservice and preservice teachers consider personal understandings and notions of recursion and induction by working through the illustrative examples and relating their own work with the stages of the model.

Illustrative Example 1 *Run to N*

Run to N is a game used with children in which a target number, *N*, is chosen, for example 20, with the following rules for playing the game.

1. Play starts with a person saying either one or two consecutive natural numbers starting with 1. For example, the first player starts by saying either "1" or "1, 2".
2. The second player has two options: (a) if the first player says "1", then the second player can say "2" or "2, 3"; (b) if the first player has said "1, 2", then the second player can say "3" or "3, 4".
3. Play continues until one player says "20" and is declared the winner.

Informal/intuitive stage

At this stage, students play the game for fun. After some play, students without prompting realize that to guarantee that they can win by saying "20", they must say "17" in their turn before that. Consider that if they have said "17", their opponent can say only "18" or "18, 19". If the opponent says "18", then the player can say "19, 20" and win. Or if the opponent has said "19", then the player can say "20" and win. Thus "17" is understood as a new target replacing 20. Eventually, students realize that smaller target numbers to guarantee a win are the following: 20, 17, 14, 11, 8, 5, 2.

The game may be varied by changing the target number and keeping the same rules. In Table 1, different target numbers are shown with intermediate target numbers for each given *N*.

Table 1: Target Numbers for Winning *Run to N*

Target number	Intermediate Target Numbers
20	17, 14, 11, 8, 5, 2
25	22, 19, 16, 13, 10, 7, 4, 1
30	27, 24, 21, 18, 15, 12, 9, 6, 3
…	…
N	Count back by 3

A variation of the game may be used if the number of consecutive numbers that can be used in play is changed. For example, suppose the players must use up to 3 consecutive numbers in the rounds of play. A different set of target numbers for this variation is seen in Table 2.

Table 2: Target Numbers for Winning *Run to N* Variation

Target number	Intermediate Target Numbers
20	16, 12, 8, 4, …
25	21, 17, 13, 9, 5 …
30	26, 22, 18, 14, 10 …
N	Count back by 4

Formal/abstract stage

In the formal/abstract stage, students realize that they are subtracting multiples of 3 in Table 1 and multiples of 4 in Table 2. This could be described in the first case of each table as 20 minus *m* multiples of 3 and 20 minus *m* multiples of 4, respectively, where *m* is the number of turns of play. And the goal in each case is to determine the smallest number that a player must say in order to win. This determination shows whether the player should play first or second in order to win.

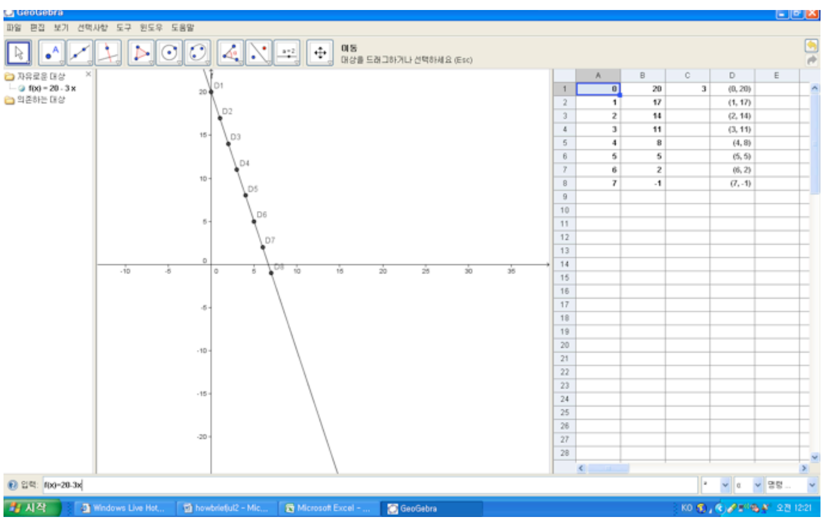
A teacher may change the game in order to challenge learners to refine strategies for winning the game. For example, the target number can be changed. Again, learners are to determine which number an individual will have to say in order to win the game.

In general, a strategy can be determined for allowing an individual to win the game. The generalization is dependent on both the target number and the number of consecutive numbers that each player must say, for example, if target number is 30 and the players must say 1 or 2 consecutive numbers, then one strategy is to step back from 30 by multiples of 3 trying to reach intermediate target numbers of 30, 27, 24, …. At this stage some learners may realize that the set of intermediate target numbers may be found using $30 - m(2 + 1)$ first where *m* is the number of the turn of play. Intermediate target numbers are seen for different target numbers in the table below.

Target number	Intermediate target numbers when one must use up to *p* consecutive numbers
20	20 – (*p* + 1), 20 – 2(p + 1), 20 – 3(*p* + 1) …
25	25 – (*p* + 1), 25 – 2(*p* + 1), 25 – 3(*p* + 1) …
30	30 – (*p*+1), 30 – 2(*p*+1), 30 – 3(*p* + 1) …
N	Count back by *p* + 1: *N* – (*p* + 1), *N* – 2(*p* + 1),…

Integrated Stage

At this stage, learners should realize that the functions generating the intermediate target number lists are linear functions. Some many need help with the suggestion that they make an ordered pair using as the *x*-coordinate the position of the intermediate target number in the list and as y-coordinate the intermediate target number in that position. For example with 30 as the target number, the pair (3, 24) represents the third intermediate target number, 24. If learners plot these ordered pairs on the coordinate plane, then they may find the equation of the linear function that passes through these points. In this way they use an algebraic graphical approach to illuminate the recursive formula. The Geogebra sheet below makes these connections.



In this series of games, now *Run to 20*, *Run to 25* and *Run to 30* with slightly modified sets of rules lead to intermediate target numbers that show young children thinking recursively in trying to decide how to win the game. This thinking is natural, born of a desire to win a game. And possibly more important from a mathematical point of view, recursive thinking is one of the few efficient ways to strategize the game successfully. This type of reasoning is used often in the study of mathematics and serves the user well.

Illustrative Example 2: ***Handshake Problem***

The handshake problem arises in a setting such as the one described below:

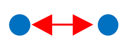
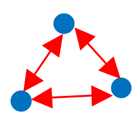
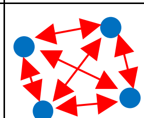
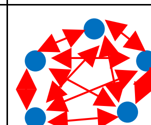
At a party, everybody shakes hands with all attendees. If there are n *people at the party and each person shakes the hand of each other person exactly once, how many handshakes take place?*

The number of handshakes can be described using different representations and a function can be found to describe the number of handshakes for a given number of participants.

Informal/intuitive stage-Moving to formal/abstract stage

In these stages, learners begin to experiment and understand that a pattern may be formed with the number of attendees and different representations may be exhibited. Such representations are seen in the following:

a. Translation of the situation to a visual model:

Number of attendees	1	2	3	4	5	6
						
Number of handshakes	0	1	3	6	10	

b. Numeric chart depicting counts

Number of attendees	Number of handshakes
1	0
2	1
3	3
4	6
5	10
6	

c. find a pattern

In these stages, students discuss the recursive process verbally:

If a third person enters, how many handshakes are added?

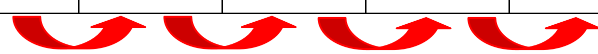
If a fourth person enters, how many handshakes are added? ...

If a tenth person enters, how many handshakes are added?

A teacher may suggest that learners answer such questions as: Could you find a rule? Could you write such a rule in your own words?

Learners may use any strategy for answering and reasoning, but they begin to find an explicit rule relating the number of attendees to the number of handshakes possibly using a recursive rule. Some may develop a table such as the one below to begin mathematizing a numeric version of the recursive rule:

Number of attendees	1	2	3	4	5	6	n
Number of handshakes	0	1	3	6	10		a_n



+ 1 + 2 + 3 + 4 ...

Learners may observe the following:

- *As the number of attendees increases by 1 in succession, the number of handshakes increases by consecutive natural numbers, +5, +6 ...*
- *The number of handshakes is the sum of the term before an attendee is added and the natural number at that stage.*

Formal/abstract recursion

In this stage, students begin to mathematize using the following:

a. Write an algebraic expression for the recursive rule, $a_n = a_{n-1} + n - 1$ where $a_1 = 0$:
Students verify the expression for the following:

Second term	$a_2 = 0 + 1 = 1$
Third term	$a_3 = 1 + 2 = 3$
Fourth term	$a_4 = 3 + 3 = 6$
Fifth term	$a_5 = 6 + 4 = 10$ …

In this stage, learners may (with teacher encouragement) realize that the recursive rule is not the only way to look at the numerical sequence describing the situation. This could be prompted by a teacher asking the learners to find the 80^{th} or 100^{th} term. Though these values can be found using the recursive model, a spreadsheet, or other devices, learners may find that it is not always efficient to use only a recursive model.

b. Some learners may observe a numerical relationship leading to an explicit function to describe the sequence. For example, using 4 as the number of attendees, learners may observe that the number a_4 of handshakes is $(3 \bullet 4)/2$, or 6.

Number of attendees	1	2	3	4	5	6	n
Number of handshakes	0	1	3	**6**	10		a_n

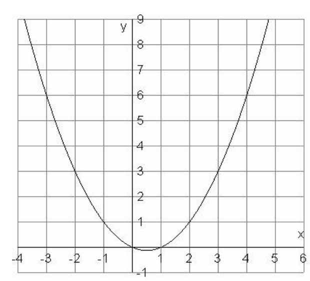
At this stage, learners generalize the function as follows: $a_n = \frac{n(n-1)}{2}$ where n is the number of attendees. At this stage, the two ways of describing the situation, the recursive rule and the explicit function, give a complete picture of the behavior of the sequence.

Integrated stage

In this stage, learners consider different representations, and different situations that fit the same algebraic model, in order to gain a deeper understanding of the problem.

a. Graphical representation

At this stage to help learners, teachers may ask students to graph the explicit function for the number of handshakes in a Cartesian plane with n as an element of the domain and a_n providing elements of the image set. And beyond that step, the domain might now lose its ties to the handshake problem and be considered as the set of all real numbers.



A teacher may also choose to extend the representation by using a spreadsheet on the function using x and $f(x)$ notation asking students to discuss the domain according to a given situation:

$$f: N \rightarrow R \text{ where } f(x) = \frac{1}{2}(x^2 - x)$$

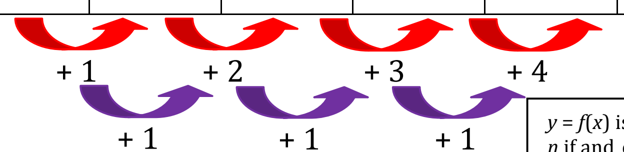
b. Analyze the graph in a given situation

It is at this stage that the learners may be encouraged to discuss a concrete or discrete situation, how it increases or decreases and the meaning of the x-axis in the situation.

c. Integrate using different strands of mathematics

1. Learners may relate the situation in discrete math to the quadratic function using continuous math.
2. They may relate the corresponding case with the Polynomial-Difference Theorem.

Number of attendees	1	2	3	4	5	6	n
Number of handshakes	0	1	3	6	10		a_n



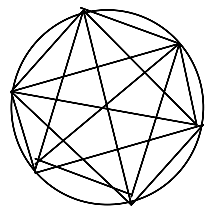
$y = f(x)$ is a polynomial function of degree n if and only if, for any set of x-values that form an arithmetic sequence, the nth differences of corresponding y-values are equal and the $(n-1)$st differences are not equal.

3. Combining the two descriptions, the recursive rule and the explicit function, a teacher may ask learners the following questions:

(a) In one case, applying these principles, there were 435 handshakes. One participant was sick and didn't come to the party. How many handshakes are missing?

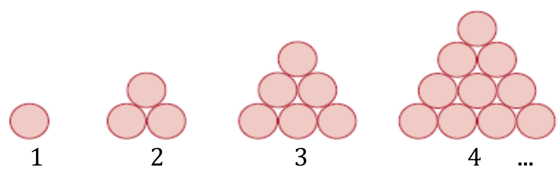
One strategy might be to have learners discover the number of participants through the explicit function, and then use the recursive rule and figure out the number of missing handshakes.

(b) There are n points on a circle. How many strings can you draw between these points? How many strings will be added with the inclusion of one more point?



One strategy is to describe the geometric problem with the same algebraic expression as the handshake problem. Each string can be seen as a handshake. This perspective may help the algebraic model become more meaningful.

c. The Triangular Numbers: In the following sequence, how many circles are in the 10th element; how many additional circles are in the 11th element?



The Triangular Numbers problem may be useful as a preparation for the rigorous level using proof by mathematical induction.

The rigorous level - induction

In this level students make the connection between the recursive rule and the explicit function, show that they describe the same process, and then generalize it for every positive integer n. One approach to the proof by mathematical induction follows.

Let S be the set of positive integers for which the recursive function and the explicit function are true where

Recursive Function: $a_1 = 0$ and $a_n = a_{n-1} + n - 1$ where n is a positive integer.

Explicit Function: $f(n) = \dfrac{n(n-1)}{2}$ where n is a positive integer.

- First check that both rules are true for positive integers 1 and 2.

Recursive Function: $a_1 = 0$ is given and $a_1 + (n - 1) = 0 + 1 = a_2$ so both 1 and 2 belong to the set *S*.

Explicit Function: When $n = 1$, then $\frac{1(1-1)}{2} = 0$ so that 1is in *S*.

Also, 2 is in *S* because $\frac{2(2-1)}{2} = 1$

- The second step is making a conjecture for any positive integer *k*, acting on the *k*th step in the same way that the a_2 step was built on the a_1 step and seeing if it is in fact the a_{k+1} step. So suppose that for some positive integer *k*, *k* belongs to set *S*. That is, for some positive integer *k*, $a_k = a_{k-1} + k - 1$ and $f(k) = \frac{k(k-1)}{2}$.

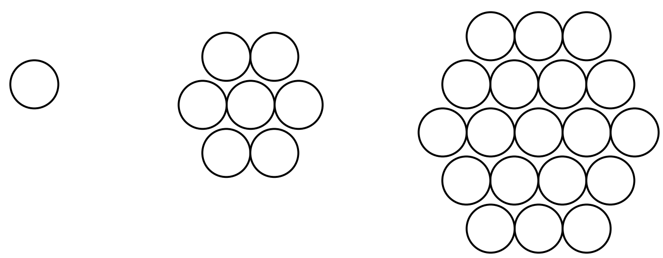
  For the recursive function, $a_k + k + 1 - 1 = a_k + k = a_{k+1}$ which implies $k + 1$ is in set *S* whenever *k* is in *S*. From these steps, we can conclude that the recursive function is true for all positive integers.

  Similarly, if for some positive integer *k*, *k* is in set *S*, then $f(k) = \frac{k(k-1)}{2}$. Then just as in the recursive argument, $f(k) + k + 1 - 1 = \frac{k(k-1)}{2} + k + 1 - 1 = \frac{k(k-1)}{2} + k = k[\frac{k-1}{2} + 1] = k[\frac{k-1+2}{2}] = \frac{k(k+1)}{2} = \frac{(k+1)(k+1-1)}{2} = f(k+1)$ so $k = 1$ is in set *S* whenever *k* is in *S* making *S* include all the positive integers and the proof is complete.

- In this proof, learners *integrate mathematics* using the two algebraic functions, recursive and explicit, and show that they both hold.
- This integration poses difficulties and misconceptions for learners because many believe that "you suppose what you have to prove". To overcome this obstacle, a teacher must be very careful with wording and language. Knowing that the functions are true for "beginning values and realizing that the function holds for some *specific value for k*, we show that the same process that allowed us to show that the function is true for two successive "beginning values in the domain" is exactly what has to happen to show that the function is true for $k + 1$ based on *k* being true. If this holds in this general case, then the proposition is true for every positive integer *n*.

Additional Problems that May Be Approached with Recursive Thinking

1. The following are the first three images in a pattern. Think of the circles as pennies. There is one penny in the first image, seven in the next. How many in the third? How many pennies would be in the next three images? How many pennies would be in the *n*th image? (Hint: You could use pennies on a table to form the next images.)



2. Consider a line with bath cloths hanging by clothespins. The first term has a bath cloth held by 2 pins; the second cloth is placed near the first so that only 3 pins are needed; the third cloth is placed near the second so that four pins are needed, and so on. How many pins are needed if this arrangement continues with 200 cloths? How do you know? Argue that your answer is correct.

3. Consider a line. Into how many regions is the line divided by *n* distinct points?

4. In a school there is a set of 21 steps. A child standing at the bottom of the steps may go upstairs taking one or two steps at a time. The question is how many different ways are there that the child can get to the top.

5. A standard European sheet of paper (first term) is designed so that when it is folded in half, the folded piece (second term) is similar to the original. When the folded piece (second term) is folded in half again, the new piece (third term) is similar to the second term, and so on. How does the area of the 100^{th} term compare to the original (first term)? How does the perimeter of the 100^{th} term compare to the original (first term)? How do you know in each case?

6. Prove the binomial theorem for any positive integer, *n*:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots$$
$$\cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

7. Into how many regions is a plane separated by *n* lines?

8. Into how many areas is space separated by *n* planes?

9. Consider the population number in year *n* (P_n) as a function of the population number in year $n - 1$ (P_{n-1}), plus the number of newborns (the rate bP_{n-1}), minus the number of deaths (using the rate dP_{n-1}) plus the net immigration (in absolute numbers). Investigate this demographic model.