Session 6, July 16
Magic Box and its Applications

1. Find the successive convergents of the continued fraction of $\frac{49}{36}$.

2. Describe (and justify, if possible) the process presented by the following table for calculating the successive convergents of the continued fraction in Problem 1.

		1	2	1	3	3
0	1	1	3	4	15	49
1	0	1	2	3	11	36

What do you notice about the 2×2 determinants?

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}, \quad \begin{vmatrix} 4 & 15 \\ 3 & 11 \end{vmatrix}, \quad \begin{vmatrix} 15 & 49 \\ 11 & 36 \end{vmatrix}$$

3. Use your work in Problem 2 to find an integral solution (x, y) of the equation $36x + 49y = 1$. Do you get the same solution as you would have gotten by back-substituting Euclid's algorithm?

4. (a) Find the GCD of 7469 and 2463.
   (b) Find the simple continued fraction for $\frac{7469}{2463}$.
   (c) What are the successive convergents of this continued fraction?
   (d) Find an integral solution (x, y) of the equation $7469x + 2463y = 1$.

5. Find the multiplicative inverse of 2689 in $\mathbf{Z}_{4001}$.

6. Find the multiplicative inverse of 4001 in $\mathbf{Z}_{2689}$.

7. Find an integral solution (x, y) of the equation $2689x + 4001y = 17$.

8. Find the multiplicative inverse of 2464 in $\mathbf{Z}_{7469}$. (Note: The numbers are not identical to Problem 4.)

9. Find an integral solution (x, y) of the equation $7469x + 2464y = 77$.

10. Find all solutions of $2464x = 11$ in $\mathbf{Z}_{7469}$. Find all solutions of $2464x = 7$ in $\mathbf{Z}_{7469}$.

11. Find the multiplicative inverse of 57 in $\mathbf{Z}_{158}$.

12. Find *all positive* integral solutions (x, y) of $158x + 57y = 2000$.

OK, enough with these Linear Diophantine equations... for now, at least. Turn to the next page for more fun.

Back to some unit group arithmetic.

13. Consider $\mathbf{U}_9 = \{1, 2, 4, 5, 7, 8\}$.
    (a) (Reminder) What was $\mathbf{U}_9$ again?
    (b) Build a multiplication table for $\mathbf{U}_9$.
14. In $\mathbf{U}_9$, What is the inverse of 4? What is the inverse of 5? What is the inverse of $4 \cdot 5$? Any conjectures?
15. Consider $\mathbf{U}_{15}$. What is the inverse of 2? What is the inverse of 7? What is the inverse of $2 \cdot 7$?
16. Explain and prove the following statment: "In $\mathbf{U}_m$, the inverse of the product is the product of the inverses."
17. In $\mathbf{U}_9$, what is the inverse of 7? What is the inverse of -7? In ordinary arithmetic, $a \cdot b = (-a) \cdot (-b)$. Does this work in $\mathbf{U}_9$? Explain.
18. Is $\mathbf{U}_9$ closed under multiplication? For any $n \in \mathbf{N}$, is $\mathbf{U}_n$ closed under multiplication? Explain.
19. Let $n \in \mathbf{N}$. Show that $\mathbf{Z}_n$ is a group under addition. Show that $\mathbf{U}_n$ is a group under multiplication.

Here are some interesting exploratory problems.

20. (Further exploration #1) Find the successive convergents of the continued fraction of $\frac{1+\sqrt{5}}{2}$. What do you notice about them?
21. (Further exploration #2) Find all solutions of $2464x = 154$ in $\mathbf{Z}_{7469}$.
22. (Further exploration #3) Prove or Disprove and Salvage if Possible: If (x_0, y_0) is an integral solution of the equation $ax + by = c$, then all integral solutions (x, y) are given by

$$x = x_0 + bt$$
$$y = y_0 - at \qquad \text{for some integer } t.$$

23. (Further exploration #4) Find three non-zero digits a_0, a_1, and a_2 and a base b such that $(a_0a_1a_2)_b = 2(a_0a_1a_2)_{10}$.