Chunks of Gold: The Best Mathematics Problems Ever Reproduced for DDPD at PCMI

Johnny W. Lott
Center for Excellence in Teaching and Learning
The University of Mississippi
Oxford, MS 38677
Phone 662/915-1395
e-mail jlott@mso.umt.edu
jlott@olemiss.edu

Warm-Ups

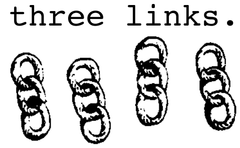
1. What is the minimum number of pitches possible for a pitcher to make a major league baseball game, assuming he plays the entire game and it is not called prior to completion?

2. Consider the following banking transaction. Deposit $50 and withdraw it as follows:

withdraw	$20	leaving	$30
withdraw	15	leaving	15
withdraw	9	leaving	6
withdraw	6	leaving	0
Total	$50	Total	$51

   Where did the extra dollar come from? To whom does it belong?

3. A businessman bought four pieces of solid-gold chain, each consisting of three links.



   He wanted to keep them as an investment, but his wife felt that, joined together, the pieces would make a lovely necklace. A jeweler charges $10.50 to break a link and $10.50 to melt it together again. What is the minimum charge possible to form a necklace using all the pieces? (Billstein, et al)

ON TO THE REAL THING

GENERAL MATHEMATICS

1. (a) Oscar developed a code in which letters were substituted for digits. Each letter in the addition given below represents one of the digits 0 through 9. What digit does each letter represent if different letters represent different digits? Is there more than one solution? (Billstein, et al)

```
                MA
                MA
              + MA
              ____
               EEL
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```
(b) FORTY          (c)  READ          (d)  GET
      TEN             - THIS             X  ON
      TEN               ____              _____
    _____               PAGE               RON
    SIXTY                                 GET
                                         ______
                                          GRAN
```

2. A census taker arrived at the door of a grizzled Montanan and asked about the number of children the woman had and their ages. The woman had little use for census takers and the government in general and said simply, "I have three sons and the product of their ages is 72. The sum of their ages is my house number." Then she stood silent. The census taker looked up at the house number and said, "That is not enough information." At this time, the woman stated, "The oldest can jump double Dutch." Then she slammed the door. The census taker went away happy. How old were the children? (Original by George Polya.)

3. A vicar and his curate, both with highly developed mathematical intelligence, were walking along a road when they saw three people coming towards them. The vicar said, "Here is a little puzzle for you, curate: The product of the ages of those people is 2,450 and

the sum is twice your age. What are their ages?" The curate thought for a few moments, and then said, "I cannot answer your question." The vicar in turn thought for a while, and then said, "Oh, yes, I see now, you are quite right of course. Well, here is one further piece of information which will enable you to solve the problem: I am older than any of those people." What is the vicar's age? (Sobel, Ed.)

4. Five sailors plan to divide a pile of coconuts among themselves in the morning. During the night, one of them decides to take his share. After throwing a coconut to a monkey to make the division come out even, he takes one-fifth of the pile. The other four sailors repeat this procedure, each throwing a coconut to the monkey and taking one-fifth of the remaining coconuts. In the morning the five sailors throw a coconut to the monkey and divide the remaining coconuts into five equal piles. What is the minimum number of coconuts that could have been in the pile originally? (Sobel, Ed.)

5. When the famous German mathematician Karl Gauss (1777-1855) was a child, his teacher required the students to find the sum of the first 100 natural numbers. The teacher expected this problem to keep the class occupied for a considerable amount of time. Gauss gave the answer almost immediately. Can you? (Billstein, et al)

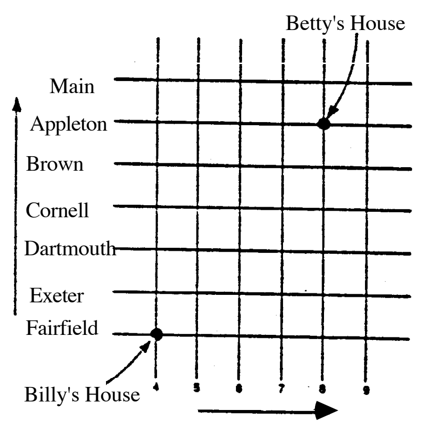
6. On the first day of a mathematics class, the teacher requested that everyone shake hands with everyone else. If there were twenty persons present and each person shakes hands just once with everyone else there, how many handshakes take place? (Billstein, et al)

7. Suppose twenty people were at a dinner party seated at a round table. Next suppose that each person shakes the hand of the person on his immediate right and left. After dinner, when the guests retire to the den, each person shakes the hand of the person on his immediate right and left. After dinner, when the guests retire to the den, each person shakes the hand of everyone with whom he has not already shaken hands. How many handshakes take place after dinner? (Billstein, et al)

8. At the end of a party, two people were amazed to discover that they had shaken the same number of hands. Should it be surprising that at any party, at least two people shake the same number of hands? (Billstein,et al)

9. How many ways are there to make change for a quarter using only dimes, nickels, and pennies? (Billstein, et al)

10. A map of a local town is shown below. Billy lives at the corner of 4th Street and Fairfield Avenue. Betty lives at the corner of 8th and Appleton Avenue. Billy decides that he will visit Betty once a day after school until he has tried every different route to her house. Billy agrees to travel only east and north. How many different routes can Billy take to get to Betty's house? (Krulik)



11. Charles and Cynthia play a game called NIM. Each has a box of matchsticks. They take turns putting 1, 2, or 3 matchsticks in a common pile. The person who is able to add a number of matchsticks to the pile to make a total of 24 wins the game. What should be Cynthia's strategy to be sure that she wins? (Billstein, et al)

12. As he grew older, Abraham De Moivre (1667-1754), a mathematician who helped in the development of probability, discovered one day that he had begun to require 15 minutes more sleep each day. Based on the assumption that he required 8 hours of sleep on date A and that from date A he had begun to require an additional 15 minutes of sleep each day, he predicted when he would die. The predicted date of death was the day when he would require 24 hours of sleep. If

this indeed happened, how many days did he live from date A? (Billstein, et al)

13. Sara and David were reading the same novel. When Sara asked David what page he was reading, he replied that the product of the page number he was reading and the next page number was 98,282. On what page was David reading? (Billstein, et al)

14. Glenn Allinger and I are willing to hire you to work for us. Our pay schemes are the following:
We will both pay you $1.00 the first day and $.50 the second day. For every day after that, you have to figure your own salary using the scheme below.
tomorrow's salary = 5/2 (today's salary) - yesterday's salary
The only differences in the rates we are willing to pay are that I always round up to the nearest dime while my compatriot always rounds down to the nearest dime. Will you work for either of us for one year? (Original by George McRae)

15. At Dunkirk, 20 boats were supposed to carry 120 soldiers across the English Channel. Some of the boats did not make it to the shore to pick up the soldiers. Instead of the 20 boats that were to carry only 6 people each, some of the boats that did make it to shore carried 6, some carried 7, and some carried 8. How many boats might have carried 7 people? (Billstein, et al)

16. Professor Noah Little designed a 24-question test on which he hoped to discourage guessing. When the test was graded, a student received 5 points for each correct answer and lost 7 points for each incorrect answer. Stu took the test, answered every question, and scored 0. How many problems did he answer correctly? (Billstein, et al)

17. Mary, a 10-year-old calculator genius, announced a discovery to her classmates one day. She said, "I have found a special five-digit number I call abcde. If I enter 1 and then the number on my calculator and then multiply by 3, the result is the number with 1 on the end." Can you find her number? (Billstein, et al)

18. In the central prison of Ilusia, there were 1000 cells numbered from 1 to 1000. Each cell was occupied by a single prisoner, and each had a separate guard. After a revolution, the new queen ordered the guards to free certain prisoners based on the following scheme. The guards walk through the prison one at a time. The first guard opens all 1000 cells. The second guard follows immediately and closes all the cells with even numbers. The third guard follows and changes every third cell starting with cell 3, that is, closing the open cells and opening the closed cells. Similarly, the fourth guard starts at cell 4 and changes every fourth cell. This process continues until the 1000th guard passes through the prison, at which point the prisoners whose cells are open are freed. How many prisoners are freed? (Billstein, et al)

19. A class from Washington School visited a neighborhood cannery warehouse. The warehouse manager told the class that there were 11,368 cans of juice in the inventory and that the cans were packed in boxes of 6 or 24, depending on the size of the can. One of the students, Sam, thought for a moment and announced that there was a mistake in the inventory. Is Sam's announcement correct? Why or why not? (Billstein, et al)

20. A group of people ordered No-Cal candy bars. The bill was $2.09. If the original price of each was 12 cents but the price has been inflated, how much does each cost? (Billstein, et al)

21. Maria finds that she has an extraordinary social security number. Its nine digits contain all numbers from 1 through 9. They also form a number such that when read from left to right, its first two digits form a number divisible by 2, its first three digits form a number divisible by 3, its first four digits form a number divisible by 4, and so on, until the complete number is divisible by 9. What is Maria's social security number? (Billstein, et al)

22. Although Euclid proved that there are infinitely many primes, it has been shown that there are strings of as many consecutive composite numbers as desired. Find 1000 consecutive natural numbers that are composite. (Billstein, et al)

23. A woman with a basket of eggs finds that if she removes the eggs from the basket either 2, 3, 4, 5, or 6 at a time, there is always one egg left. However, if she removes the eggs 7 at a time, there are no eggs left. If the basket holds up to 500 eggs, how many eggs does the woman have? (Billstein, et al)

24. When the marchers in the annual Mathematics Department Parade lined up 4 abreast, there was 1 odd person, when they tried 5 in a line, there were 2 left over; and when 7 abreast, there were 3 left over. How large is the Department? (Dudley)

25. "On the twelfth day of Christmas, my true love sent to me twelve drummers drumming, eleven pipers piping, ten lords a-leaping, nine ladies dancing, eight maids a-milking, seven swans a swimming, six geese a-laying, five gold rings, four calling birds, three French hens, two turtle doves, and a partridge in a pear tree."
Suppose your true love had given you all the gifts in the above song by December 25. That evening the two of you had a disagreement and you felt bitter. If you returned the gifts one each day starting December 26, when would you finish? Suppose the next year is not a leap year. (Newell)

26. What do you know about Pythagorean triplets?

27. Factors of a locker number are 2, 5, and 9. If there are exactly nine other factors, what is the locker number? (Billstein, et al)

28. Find the remainder when 3 raised to the 100th power is divided by 5. (Billstein, et al)

29. Candy bars priced at 50 cents each were not selling, so the price was reduced. Then they all sold in one day for a total of $31.93. What was the reduced price for each candy bar? (Billstein, et al)

30. A watermelon weighing 100 pounds was found to be 99% water. After sitting in the sunlight all day, some of the water evaporated, leaving the melon 98% water. How much did the melon weigh after the evaporation occurred? (Billstein, et al)

31. The crust of a certain pumpkin pie is 25% of the pie. By what percent should the amount of crust be reduced in order to make it constitute 20% of the pie? (Billstein, et al)

32. What is the least number of weights that can be used in order to be able to weigh any amount from 1 ounce to 680 ounces on a balance scale? (Billstein, et al)

33. A computer is programmed to scan digits of successive integers. For example, if it scans the integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, it has scanned 15 digits. How many digits has the computer scanned if it scans the consecutive integers from 1 through 10,000,000? (Billstein, et al)

34. Without using a calculator, find the number of digits in the number 2^12*5^8. (Billstein, et al)

35. Who will get the nomination? Each person on the checkerboard is a candidate for the presidency. The object is to remove eight people, leaving one person on the center square. This is to be done in the fewest possible number of moves. A move consists of (1) moving a person to any adjacent square, up and down, left and right, or diagonally; (2) jumping a man as in checkers except that the jump can also be up and down, left and right or diagonally. The jumped person is then removed. To work on the puzzle, put buttons or coins of the nine persons. (Gardiner, Ed.)

36. A person sold n cows for $n per cow. With the proceeds, an odd number of sheep were bought for $10 each, and a pig was bought for less than $10. How much did the pig cost? (Dudley)

37. How many different ways can thirty nickels, dimes, and quarters be worth $5? (Dudley)

38. The following problem is at least 400 years old: Find the number of men, women, and children in a company of 20 if together they pay $20, each man paying $3, each woman $2, and each child $.50. (Dudley)

39. Anna says, "We three have $100 altogether." Betty says, "Yes, and if you had six times as much and I had one-third as much, we three would still have $100." Carl says, It's not fair. I have less than $30." Who has what? (Dudley)

40. Anna took 30 eggs to market and Barbara took 40. Each sold some of her eggs at 5 cents per egg and later sold the remainder at the same lower price (in cents per egg). Each received the same amount of money. What is the smallest amount that they could have received? (Dudley)

41. In a certain casino, red chips sell for $7 each, blue chips sell for $6 each, and white chips sell for $5 each. A disagreeable loser came into the casino and said, "Give some red chips, half as many blue chips, and some white chips. Here's $40. Keep the change--there won't be any." What did the person get? (Dudley)

LOGIC

1. A pioneer moving west had a goose, a bag of corn, and a fox. He came to a river. The ferry was large enough to carry him and one of his possessions. If he were to leave the fox and the goose alone, the fox would eat the goose. If he were to leave the goose and corn alone, the goose would eat the corn. How did he get himself and his possessions across the river?

2. Arrange the four aces, four kings, four queens, and four jacks of an ordinary bridge deck so that there is exactly one ace, king, queen, and jack in each row, column, and diagonal of a 4x4 square and so that there

is exactly one spade, heart, diamond, and club in each row, column, and diagonal of the square. (Lott)

3. At the end of a tour of the Grand Canyon, several guides were talking about the people on the latest British-American tour. The guides could not remember the total number in the group; however, together they compiled the following statistics about the group. It contained 26 British females, 17 American women, 17 American males, 29 girls, 44 British citizens, 29 women, and 24 British adults. Find the total number of people in the group. (Billstein, et al)

4. A paper carrier delivers 31 copies of the Town Gazette and 37 copies of the Daily Flyer each day to 60 houses. If no house received 2 copies of the same paper, answer the following:
What is the least number of houses to which 2 papers could have been delivered?
What is the greatest number of houses to which 2 papers could have been delivered? (Billstein, et al)

5. Howie, Frank, and Dandy each tried to predict the winners of Sunday's professional football games. The only team not picked that is playing Sunday was the Giants. The choices for each person were as follows:
Howie: Cowboys, Steelers, Vikings, Bills
Frank: Steelers, Packers, Cowboys, Redskins
Dandy: Redskins, Vikings, Jets, Cowboys
If the only team playing Sunday are those just mentioned, which teams will play which other teams? (Billstein, et al)

6. Suppose you and I have the same amount of money. How much must I give you so that you have ten dollars more than I? (Smullyan)

7. A certain convention numbered one hundred politicians. Each politician was either crooked or honest. We are given the following two facts:
(1) At least one of the politicians was honest.
(2) Given any two of the politicians, at least one of the two was crooked.
Can it be determined from these two facts how many of the politicians were honest and how many were crooked? (Smullyan)

8. A dealer bought an article for $7, sold it for $8, bought it back for $9, and sold it for $10. How much profit did he make? (Smullyan)

9. Many of you are familiar with Frank Stockton's story "The Lady or the Tiger?," in which the prisoner must choose between two rooms, one of which contains a lady and the other a tiger. If he chooses the former, he marries the lady; if he chooses the latter, he (probably) gets eaten by the tiger.
The king of a certain land had also read the story, and it gave him an idea. "Just the perfect way to try my prisoners!" he said one day to his minister. "Only, I won't leave it to chance; I'll have signs on the doors of the rooms, and in each case I'll tell the prisoner certain facts about the signs. If the prisoner is clever and can reason logically, he'll save his life--and win a nice bride to boot!"
THE FIRST TRIAL
One of the following signs is true.
1. In this room there is a lady and in the other room there is a tiger.
2. In one of these rooms there is a lady, and in one of these rooms there is a tiger.
THE SECOND TRIAL
Now both signs are true or both signs are false.
1. At least one of these rooms contains a lady.
2. A tiger is in the other room.
THE THIRD TRIAL
Again either both signs are true or both false.
1. Either a tiger is in this room or a lady is in the other room.
2. A lady is in the other room.
(Smullyan)

10. Two cyclists started riding their bikes at 9:00 a.m. at City Hall. They followed the local bike trail and returned to City Hall at the same time. However, Jose rode three times as long as Lucimar rested on her trip and Lucimar rode 4 times as long as Jose rested on his trip. Assuming that each cyclist rode at a constant speed, who rode faster? (Billstein, et al)

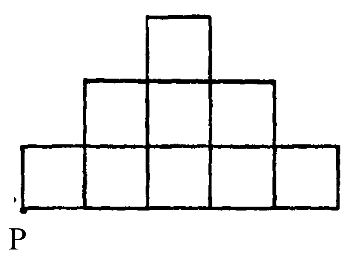
11. Find a number ending in 7 which is multiplied by 5 when the 7 is moved to the left of the other digits. (Sobel, Ed.)

ALGEBRA

1. Big Scare Airlines allows each passenger to carry x pounds of luggage free of charge, with an additional charge for each extra pound. The combined weight of luggage for Mr. and Mrs. Byrd was 105 pounds. Mr. Byrd and Mrs. Byrd had to pay $1.00 and $1.50, respectively for extra weight. They then noticed that a third passenger also had 105 pounds of luggage and was charged $6.50 for the number of pounds over the x-pound limit. How many pounds are allowed for each passenger without a charge? (Billstein, et al)

2. A publishing company hires two proofreaders, Al and Betsy, to read a manuscript. Al finds 48 errors and Betsy finds 42 errors. The editor finds that 30 common errors were found by the proofreaders; that is, 30 errors were found by both Al and Betsy. What is your estimate of the number of errors not yet found? (Billstein, et al)

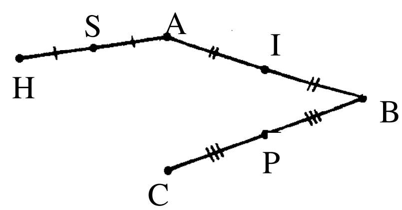
3. A rancher purchases a plot of land surrounded by a fence. The former owner had marked off nine squares of equal size to subdivide the land, as shown below. The rancher wants to divide the land into two plots of equal area. To divide the property, he wishes to build a single, straight fence beginning at the far left corner (point P on the drawing). Is such a fence possible? If so, where should it be? (Billstein, et al)



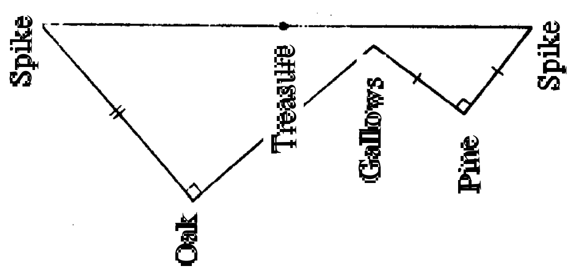
4. A school committee meeting began between 3:00 and 4:00 p.m. and ended between 6:00 and 7:00 p.m. The positions of the minute hand and the hour hand of the clock were reversed at the end of the meeting from what they were at the beginning of the meeting. When did the meeting start and end? (Billstein, et al)

5. One day Linda left home H for school S. Rather than stopping at school, she went on to the corner A which is twice as far from home as the school and on the same street as her home and the school. Then she

headed for the ice cream parlor I. Passing the ice cream parlor, she headed straight for the next corner B, which is on the same street as A and I and twice as far from A as I. Walking toward the park P, she continued beyond it to the next corner, C, so that C, P, and B are on the same street and C is twice as far from B as P. At this point, she again headed for the school but continued walking in a straight line twice as far, reaching point D. Then she headed for the ice cream parlor, but continued in a straight line twice as far to point E. From E, Linda headed for the park but continued in a straight line twice as far to F, where she stopped. The figure below shows the first part of Linda's walk. What is the location of Linda's final stop? (Billstein, et al)



6. Among his great-grandfather's papers, Jose found a parchment describing the location of a hidden treasure. The treasure was buried by a band of pirates on a deserted island that contained an oak tree, a pine tree, and a gallows where the pirates hanged traitors. The map looked like the one below and gave the following directions.



Count the steps from the gallows to the oak tree. At the oak, turn 90 degrees to the right. Take the same number of steps and then put a spike in the ground. Next, return to the gallows and walk to the pine tree, counting the number of steps. At that pine tree, turn 90 degrees to the left, take the same number of steps, and then put another spike in the ground. The treasure is buried halfway between the spikes. Jose found the island and the two trees but could not find the gallows or the spikes, which had long since rotted. Jose dug all over the island, but because the island was large, he gave up. Devise a plan to help Jose find the treasure. (Billstein, et al)

7. The queen of Ilusia lost a war and was forced to divide her queendom into a number of smaller countries, no two the same size. The queen also had to distribute all 1000 gold bars in her treasury to the countries in such a way that each country received 1 bar more than the next smaller country. No bars could be broken. If the queen divided her queendom into the least number of countries possible and divided her gold as described, how many countries were formed and how many bars of gold did each country receive? (Billstein, et al)

8. A woman bought a dozen pieces of fruit--apples and oranges--for 99 cents. If an apple costs 3 cents more than an orange, and she bought more apples than oranges, how many of each did she buy? (Dudley)

9. A man cashes a check for d dollars and c cents at a bank. Assume that the teller by mistake gives the man c dollars and d cents. Assume that the man does not notice the error until he has spent 23 cents. Assume further that he then notices that he has 2d dollars and 2c cents. Assume still further that he asks you what amount the check was for. Assuming that you can accept all the assumptions, what is the answer? (Dudley)

10. A ball bounces 0.8 of the distance from which it is dropped each time it is dropped. If it is dropped from a height of 6 feet, what is the least number of bounces the ball makes before it rises to a height of less than 1 foot? (Billstein, et al)

11. Jim has saved some silver dollars. He wants to divide them among Tom, Dick, Mary, and Sue so that Tom gets 1/2 of the total amount, Dick gets 1/4, Mary gets 1/5, and Sue gets 9 of the dollars. How many dollars has Jim saved? (Billstein, et al)

12. Willie can row 4 miles per hour in still water. If it takes him 2 hours to cover a certain distance rowing upstream and only 1 hour to row the same distance downstream, how fast is the current moving? (Billstein, et al)

13. Ann and Sue bought identical boxes of stationery. Ann used hers to write 1-sheet letters and Sue used hers to write 3-sheet letters. Ann used all the envelopes and had 50 sheets of paper left, and Sue used all the sheets of paper and had 50 envelopes left. Find the number of sheets of paper in each box. (American High School Mathematics Examination, 1972)

14. Find the sum of all the digits in the integers 1 through 1,000,000,000. (Billstein, et al)

15. With 400 members voting, the House of Representatives defeated a bill. A revote, with the same members voting, resulted in passage of the bill by twice the margin with which it was originally. How many more members voted for the bill the second time than voted for it the first time? (Annual High School Mathematics Examination, 1968)

16. A man has walked two-thirds of the distance across a railroad bridge when he sees a train approaching, at 45 miles per hour. If he can just manage to escape by running at uniform speed to either end of the bridge, at what rate of speed must he run to avoid the train? (Sobel, Ed.)

17. Solve for x and y:
$1x + 2y = 3$
$4X + 5y = 6$
Generalize the above using two unknowns. Does the above generalize to three dimensions? (From a class by Avital and Libeskind in 1977)

PROBABILITY

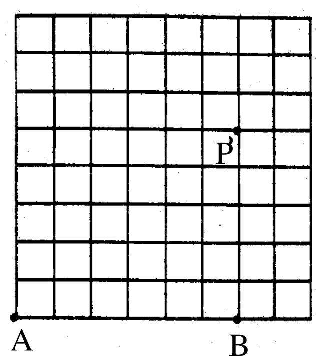
1. A butcher wrapped three 1-pound packages of meat in butcher paper while having a conversation with a customer. The packages contained round steak, ground beef, and sausage and were to be labeled R, G, and S, respectively. During the conversation, the butcher forgot which package was which, but labeled them anyway. What is the probability that each of the packages was labeled correctly? (Billstein, et al)

2. Jane has two tennis serves, a hard serve and a soft serve. Her hard serve has a 50% chance of being good. If her hard serve is good, then she has an 80% chance of winning the point. Her soft serve has a 90% chance of being good. If her soft serve is good, she has a 50% chance of winning the point.
(a) What is the probability that Jane wins the point if she serves hard and then, if necessary, soft?
(b) What is the probability that Jane wins the point, if she serves hard and then, if necessary, hard?
(Billstein, et al)

3. A woman in a small foreign town applies for a marriage permit when she is 18. To obtain the permit, she is handed six strings that she must hold in her hand so that the ends of the strings are exposed. On one side, the ends (top or bottom) are picked randomly, two at a time, and tied, forming three separate knots. The same procedure is then repeated for the other set of string ends, forming three more knots. If the tied strings form one closed ring, the woman obtains the permit. If not, she must wait until her next birthday to reapply for a permit. What is the probability that she will obtain a marriage permit on her first try? If a woman fails to get a ring 10 years in a row, she must remain single. What is the probability of such a streak of bad (good) luck? If the number of strings were reduced to three and the rule was that an upper end must be tied to a lower end, what is the probability of a single ring? If the number of strings were three, but an upper end could be tied to either an upper or lower end, what is the probability of a single ring? What is the probability of forming three rings in the original problem? What is the probability of forming two rings in the original problem? (Billstein, et al)

4. Al and Betsy were playing a coin-tossing game in which a fair coin was to be tossed until a total of either three heads or three tails occurred. Al was to win when a total of three heads were tossed and Betsy was to win when a total of three tails were tossed. Each bet $50 on the game. The coin was lost when Al had two heads and Betsy had one tail. How should the stakes be fairly split if the game is not continued? (Billstein, et al)

5. In a portion of a large city, the streets divide the city into square blocks of equal size, as shown below. A taxi driver drives daily from point A to P. One day she drove from A to B along AB and the from B to P along BP. If she does not want to cover any distance longer than AB + BP, how many possible routes are there from A to P? (Billstein, et al)



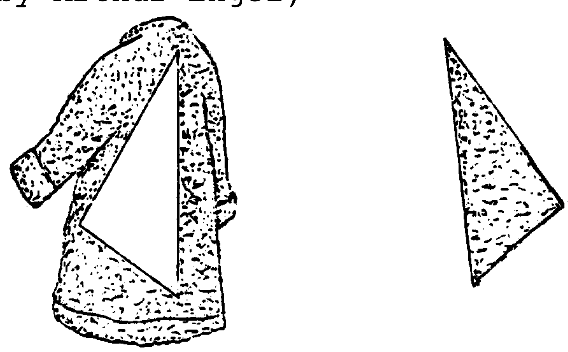
6. During the baseball season thus far, Reggie has been at bat 381 times and has a batting average of approximately 0.291--that is, his number of hits divided by 381 is 0.291. If he has 60 more official times at bat this season, how many hits must he get in order to end the season with a batting average of 0.300 or better? (Billstein, et al)

7. Show that the thirteenth day of the month is more likely to be Friday than any one of the other days of the week. (Sobel, Ed.)

GEOMETRY

1. If you are given a set of unit squares and a given area, what is the minimum and maximum perimeters that can be determined with the following rules for acceptable figures to use. (*Student Math Notes* by Judy Mumme)

2. In Sikinia, people are very poor, but everyone owns a ferocious dog. These dogs tear triangular holes into the clothes of passers-by. One expensive breed of dogs tears holes in the shape of perfect squares. The victims don't throw their clothes into trash cans. They go to a tailor to mend them. There was a poor tailor who was making a living patching up holes. When the hole was a square he cut a square piece of cloth. To test it he would fold it along each diagonal. Is this a good test? Is there a folding test for a square? The secret dream of our poor tailor was to become rich by mending mink coats. One day he had his big chance. A lady came in with a mink coat which had a huge triangular hole on the back as below. Our poor tailor had never mended furs before, but only regular cloth. And he made a tragic mistake. On mink, hair grows on one side only. The other side is clean shaven. You cannot turn it over like cloth which looks the same on both sides. But our poor tailor had to learn this the hard way. He cut a patch to fit the hole, but it fit only on the wrong side. What to do now? How can we help our poor tailor? (Original by Arthur Engel)

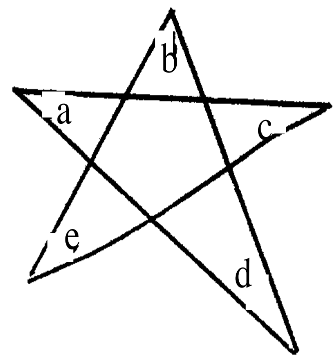


[Hint: Symmetric pieces can be turned over. So we must cut up the patch into symmetric pieces. But you should do it in as few cuts as possible.]

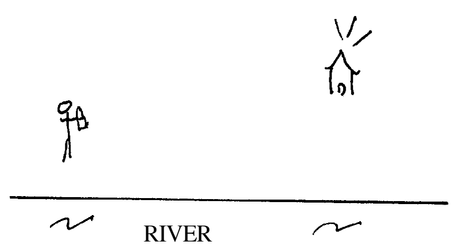
3. Jernigan, a stained-glass window maker, was asked by a customer to cut a convex polygonal piece of glass in which the smallest interior angle is 120 degrees and each successive angle is 5 degrees greater than its predecessor. Can he make such a construction, and if so, how many sides will the piece of glass have? (Billstein, et al)

4. What is the maximum number of collinear vertices that an n-gon may have? (Lott)

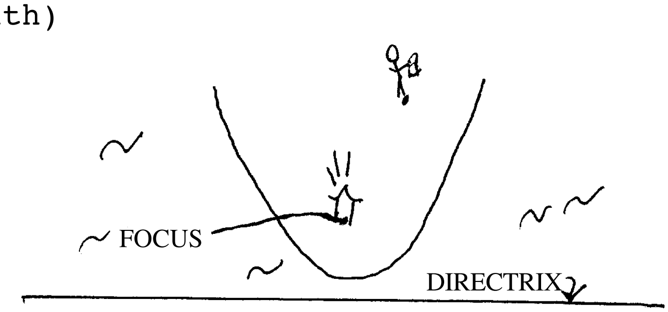
5. Find the sum of the measures of the angles, a, b, c, d, and e, of any five-pointed star as the one pictured below. What is the sum of the measures of the angles in any seven-pointed star? (Hint: this is a loaded question. (Billstein, et al and discussion from *Student Math Notes* by Maurice Burke)



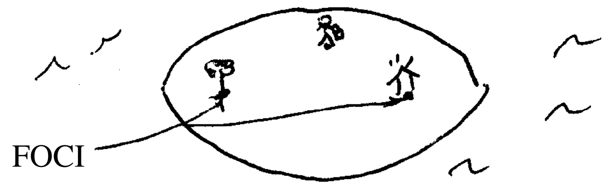
6. A hiker, carrying a bucket, sees that his tent is on fire. To what point on the bank of the river should the hiker run to fill his bucket in order to make his trip to the tent as short as possible. (Lott and Smith)



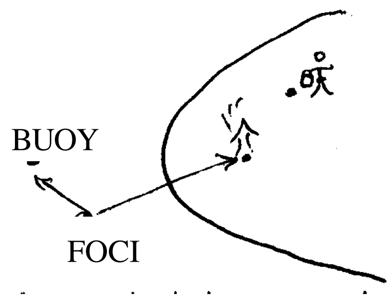
7. Suppose the hiker is on Parabolic Peninsula as pictured below, and the same question is posed. (Lott and Smith)



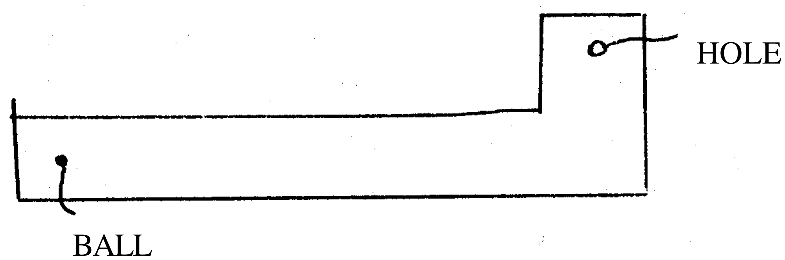
8. Suppose the hiker is on Elliptic Isle as pictured below and the same question is posed. (Lott and Smith)



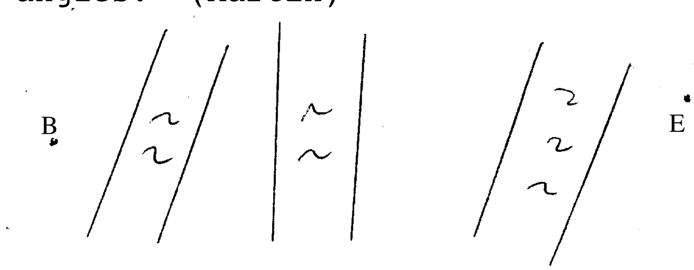
9. Suppose the hiker is on Hyperbolic Bay as pictured below and the same question is posed. (Lott and Smith)



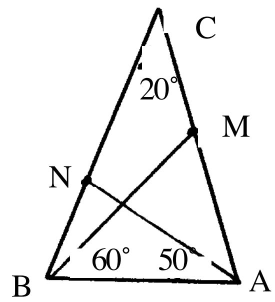
10. In the figure below, sketch a construction of a path of a hole-in-one in the miniature golf course. (Martin)



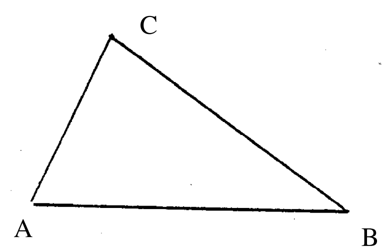
11. Sketch the shortest path from B to E that crosses each of the three indicated "rivers" in the figure below at right angles. (Martin)



12. In isosceles triangle ABC below, find the measure of the angle marked x. (Sobel, Ed.)

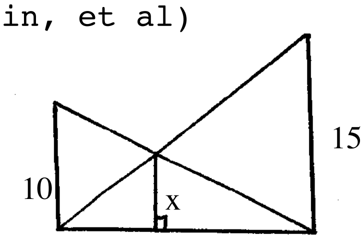


13. Given acute triangle ABC, construct the square inscribed in the triangle that has a side on AB. (Martin)



14. Two circles are concentric. A tangent to the inner circle forms a chord 12 inches long in the larger circle. Find the area of the ring between the circles. (Sobel, Ed.)

15. Two neighbors, Smith and Wesson, planned to erect flagpoles in their yards. Smith wanted a 10-foot pole, and Wesson wanted a 15-foot pole. In order to keep the poles straight while the concrete bases hardened, they agreed to attach the guy wires from the tops of the flagpoles to a fence post on the property lines and to the bases of the flagpoles as shown below. How high should the fence post be and how far apart should they erect flagpoles for this scheme to work? (Billstein, et al)



16. Mr. Lopez, who is 6 feet tall, wants to install a mirror on his bedroom wall that will enable him to see

a full view of himself. What is the minimum-length mirror that will serve his needs, and how should it be placed on his wall? (Krulik)

17. A wooden cube whose edges are 3 inches long is to be cut into 27 one-inch cubes. If, after each cut with a saw, the resulting pieces may be lined up or piled up and the next cut made through the pile, what is the smallest number of cuts that will provide the desired dissection? (Sobel, Ed.)

18. A bridge that spans a bay is 1 mile long and is suspended from two supports, one at each end. As a result, when it expands a total of 2 feet from the summer heat, it "buckles" in the center, causing a bulge. How high is the bulge? (Krulik)

19. A quadrilateral has a right angle and the two sides not adjacent to that angle are each 6 inches long. What is the maximum area and what shape will this area have? (Sobel, Ed.)

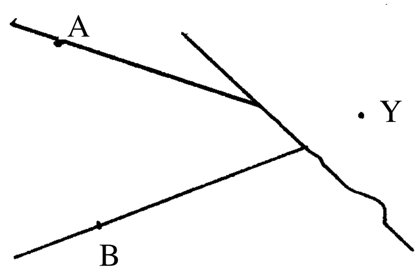
20. Three cylindrical oil drums of 2-foot diameter are to be securely fastened in the form of a "triangle" by a steel band. What length of band will be required? (Krulik)

21. There once was a couple, Adam and Eve. They had a son Cain. To develop Cain's intellect, Adam likes to pose problems. One day he showed Cain a triangular pie, and he said, "You may choose any point 0 in the plane of pie and reflect through 0 to an image. The intersection of the original and the image is yours. Answer the following questions.
    (a) Cain got nothing. Where did he choose 0?
    (b) Cain got a piece in the shape of a parallelogram. Where did he choose 0?
    (c) Cain got a hexagon. Where did he choose 0? What shape is the hexagon?
    (d) Can he get a triangular piece of pie?
    (e) What else can he get?
    (f) Where should he choose 0 to get as much as possible?
    (g) For what shapes of the pie can Cain get all of it?
    (h) The pie was a polygon and he managed to get all of it. Deduce as many properties of the polygon as you can. (Original by Arthur Engel)

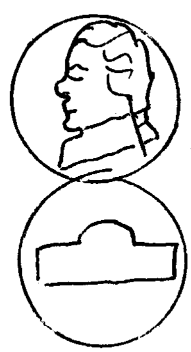
22. A manufacturer of metal cans has a large quantity of rectangular metal sheets 20 cm wide by 30 cm long. Without cutting the sheets, the manager wants to make cylindrical pipes with circular cross sections from some of the sheets and box-shaped pipes with square cross sections from the other sheets so that the volume of the box-shaped pipes is greater than the volume of the cylindrical pipes. Is this possible? If so, how should the pipes be made and what are their volumes?

23. Suppose a wire is stretched tightly around the earth. (The radius of the earth is approximately 6400 km.) If the wire is cut, its circumference increased by 20 m and the wire is then placed back around the earth so that the wire is the same distance from the earth at every point, could you walk under the wire? (Billstein, et al)

24. The vertex of angle AVB is obstructed in the figure below. Without using the region behind the obstruction, construct that part of the angle bisector that is not behind the obstruction. (Martin)



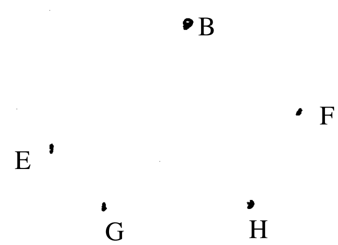
25. If the heads up coin is rolled around the tails up coin in the figure below, until the heads up coin is directly under the other, will the head then be upside down? (Martin)



26. David is serving a fresh apple pie to his three brothers and himself. His older brother, Mike, bets his share that David cannot cut the pie into four equal portions without lifting his knife and/or going back over a cut. David thinks for a minute and then wins the bet. How does he do this? (Krulik)

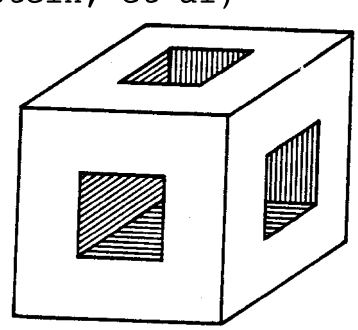
27. A cube that is 3 inches on each edge is painted completely red on all 6 faces. The cube is then cut into 27 smaller cubes, each measuring 1 inch on each edge. Of these 27 smaller cubes, how many have exactly 3 faces painted red? How many have exactly 2 faces painted red? How many have exactly 1 face painted red? How many have no faces painted red? (Krulik)

28. In the figure below, sketch a pentagon having points E, B, F, H, G as midpoints of the sides taken in order. (Martin)

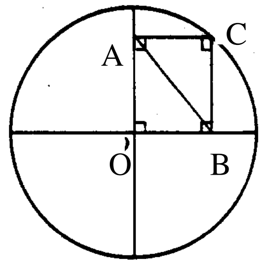


29. Is a tennis ball can larger around, or taller, or are both of these measures the same?

30. A toy maker decides to design a wooden cube with square holes in each of the cube's faces. The holes extend all the way through the cube, as shown below. The toymaker wants the length of each side of the square holes to be one-third the length of a side of the cube. If the total surface area of the toy is to be 2 square meters, how long should the sides of the cube be? (Billstein, et al)



31. In the circle below, two perpendicular diameters are shown. From point C on the circle, two segments AC and CB are drawn so that quadrilateral ACBO is a rectangle. If the diameter of the circle is 10 cm, find the length of AB. (Billstein, et al)



32. A race track is 400 feet along the straight section and has a semicircle 50 feet at each end. If one horse runs 10 feet from the inside railing at all times and a second horse runs 20 feet from the railing at all times, what is the difference in distance covered by the two horses in one lap of the track? (Billstein, et al)

33. A circular object 1/2 inch in diameter is dropped on a large grid consisting of squares that measure 3 inches on a side. If the circular object falls within the grid, what is the probability that it will not touch a grid line? (Billstein, et al)

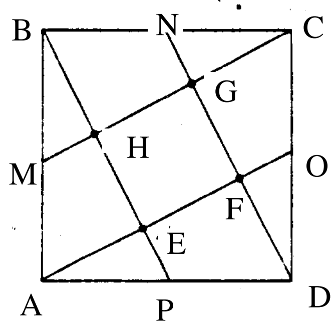
34. Starting at the same time from diametrically opposite points, Linda and David travel around a circular track at uniform speeds in opposite directions. They meet after David has traveled 100 m and a second time 60 m before Linda completes one lap. Find the circumference of the track. (American High School Mathematics Examination, 1970)

35. How many squares in a square? rectangles in a rectangle? triangles in a triangle? (Lott)

36. Given two equilateral triangles, find an equilateral triangle whose area is the sum of the areas of these two triangles. (Brown and Walter)

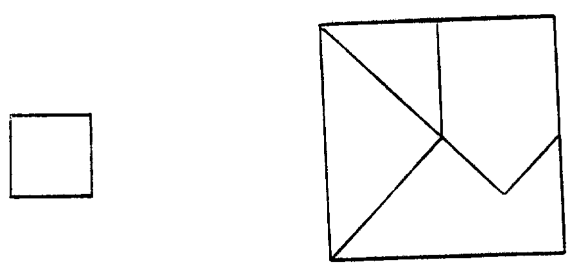
37. Lines from the vertices of square ABCD to the midpoints of the sides M, N, O, and P are shown in the following figure.
(a) Prove that quadrilateral HGFE is a square.

(b) What is the area of square HGFE if the area of quadrilateral ABCD is 100 square centimeters? Justify your answer. (Billstein, et al)

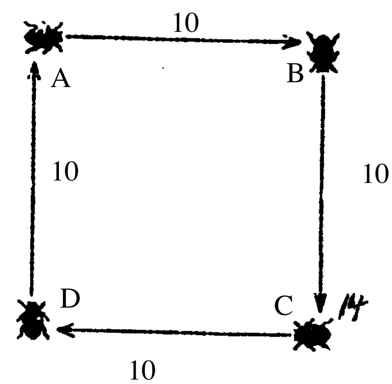


38. Can a solid cube one foot on a side be constructed from bricks 2" x 4" x 8"? You must prove your answer. (Borrowed from Howard Reinhardt, an old friend)

39. Pretend you are Pythagoras. Show that the sum of the two squares sketched below equals a square. You are to cut out each of these squares and then cut the larger square into 4 smaller pieces as marked. Then reassemble the resulting five pieces into a square. Is it really possible?



40. Four Bugs--A, B, C, and D--occupy the corners of a square 10 inches on a side as below. A and C are male, B and D are female. Simultaneously A crawls directly toward B, B toward c, C toward D and D toward A. If all four bugs crawl at the same rate, they will describe four congruent logarithmic spirals which meet at the center of the square. How far does each bug travel before they meet? (The problem can be solved without calculus.) (Gardiner, 1959)



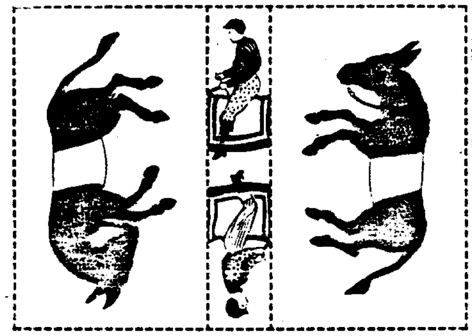
41. A cylindrical hole six inches long has been drilled through the center of a solid sphere. What is the volume remaining in the sphere? (Gardiner, 1959)

MISCELLANEOUS

1. In a Jewish revolt against Rome, Josephus and 39 of his comrades were holding out against the Romans in a cave. With defeat imminent, they resolved that, like the rebels at Masada, they would rather die than be slaves to the Romans. They decided to arrange themselves in a circle. One man was designated as number one, and they ?? different in other versions of the tale). Now it is obvious whose turn it was to be killed on the first time around: numbers 7, 14, 21, 28, and 35. From this point on, the outcome is not as clear. Josephus figured out instantly where he ought to sit in order to be the last to go. But when the time came, he joined the Roman side instead of killing himself. He lived to write his famous histories: The Antiquities and The Jewish War. Where did he seat himself?

2. A worm is at one end of a 1 km rubber rope. The worm crawls along the rope at a steady rate of 1 cm per second. After the first second, the rope stretches like a rubber band to 2 km. After the next second, it stretches to 3 km, and so on. How long does the worm take to reach the end of its rope? How long will the rope be when it does reach the end? (Hint: Try a simpler version of the problem when the rope is 5 cm long.) (Carey from many years ago.)

3. Cut on the dotted lines; reassemble so the riders are mounted on the horses. (Folding is not allowed) (Gardiner)



4. Into how many pieces can you cut a cheese with one cut of a knife? with two cuts? three? four? ...Moving pieces following a cut is not allowed.

5. How many hexagons and pentagons are in a soccer ball?

6. A classroom has 5 rows of 5 desks per row. The teacher requests the 25 students to change seats so that each student moves to an adjacent seat. Is it possible to obey the teacher?

7. Given an integer, is there always a multiple of it which has only 0s and 1s as digits?

8. Two travelers spend from 3 o'clock till 9 in walking along a level road, up a hill, and home again: Their pace on the level being 4 miles an hour, up hill 3, and down hill, 6. Find the distance walked: also within half an hour, time of reaching top of hill.

9. Using only the irrationality of the square root of 2 show that there exist irrational numbers a and b such that a raised to the b power is a rational number. (This is Gelfond's problem.)

10. You are given 12 coins of the same denomination; one of them is counterfeit. The counterfeit coin has different weight than the other coins. Using a balance and 3 weighings only, you are to determine which coin is counterfeit and whether it is lighter or heavier than the other 11 coins. (Carey)

Integrated Mathematics

Problems below are taken from ideas from the SIMMS IM materials.

1. According to a newspaper report, the trees in a certain land area are being cut at a rate of 15% per year. The lumber company claims that it replants 2000 trees every year in this area. Discuss the future tree production of this land area if this plan continues.

2. Given a box chosen by you, find the percent of waste that a company would have when it stamps out the net of the box from a roll of cardboard used to make boxes of this type.

3. Consider two sets of maps of Greenland and Mauritania from an atlas. Why could Greenland be bigger than Mauritania in one and smaller in another?

4. How can I fairly divide a grandfather clock between two people?

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