7 *Sweet Child O' Mine*

Important Stuff

We were going to call today's set "Children of the Corn" for linear algebra fanatics, a nod to kernel reference. That would've been a-maize-ing.

PROBLEM

Pick one or more of the following equations. Coordinate with those at your table so that not everyone picks the same equation.

- $4x = 8$
- $x^2 + x = 2$
- $x^2 = 1$
- $x^2 = 2x$
- $x^2 + 2x = 3$
- $x^3 + 2x + 3 = 0$

It's time for some x-coordination!

On the array of numbers from 0 to 59 below...

1. Draw a circle around all of the numbers that are solutions to your equation in mod 5. That means if you circled 2, you should also circle 7, 12, 17...
2. Draw an X through all of the numbers that are solutions to your equation in mod 12. That means if you X'ed 2, you should also X 14, 26...
3. Find all solutions to your equation in mod 60.

Instead of using circles and X's, you could also use markers! (Darryl: Polars!) Yes, he actually said this.

0	1	2	3	4	5	6	7	8	9	10	11
12	13	14	15	16	17	18	19	20	21	22	23
24	25	26	27	28	29	30	31	32	33	34	35
36	37	38	39	40	41	42	43	44	45	46	47
48	49	50	51	52	53	54	55	56	57	58	59

Let $f(n)$ be the number of solutions to your equation in mod n, remembering that the actual numbers in mod n go from 0 to $n - 1$. What is $f(5)$? What is $f(12)$? What is $f(60)$?

Make sure $f(5)$ isn't larger than 5... that would be unpossible!

1. A number is 3 more than a multiple of 5, and 7 more than a multiple of 12. Find some possible values of this number.

2. $\phi(n)$ counts the number of values between 1 and n that are relatively prime to n.

(a) What is $\phi(5)$? What is $\phi(12)$?

(b) If a number shares no common factors with 5 *and* no common factors with 12, what can you say about it?

(c) Use the method from today's box problem, or the method presented yesterday, to find all the numbers that are relatively prime to 60.

Two numbers are *relatively prime* if they share no common factors greater than 1. It's just a quick way of saying that longer phrase.

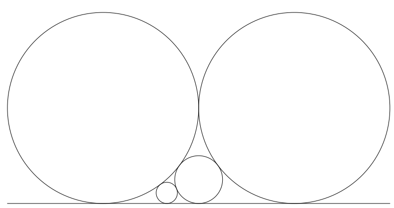
1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60

(d) What is $\phi(60)$?

3. Let $P(n)$ be defined as the number of solutions to $xy = 1$ in mod n. What is $P(5)$? What is $P(12)$? What is $P(60)$?

Actually tracking down all the solutions making up $P(60)$ might take a while, so perhaps you could find a faster way!

4. Two circles with diameter 1 are tangent to the x-axis, and mutually tangent at the point $(\frac{1}{2}, \frac{1}{2})$. Two smaller circles are packed in as shown below so they are tangent to the axis and the other circles. Use the overlaid grid on the handout to find the diameters and the coordinates of the centers of each of these smaller circles.



Lots of circles and X's today. Matt and Debra have abandoned the problem set in favor of tic-tac-toe.

5. Fill in these two grids; one is the set of all possible sums when rolling two dice, and the other is the piece-by-piece expansion of $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$.

+	1	2	3	4	5	6
1						
2			5			
3						
4						
5		7				
6						12

$\times$	x	x^2	x^3	x^4	x^5	x^6
x						
x^2			x^5			
x^3						
x^4						
x^5		x^7				
x^6						x^{12}

Ahem! In Gema's home state of Texas, these are called *number cubes*! Get it right!

Cool, huh?

6. Use technology to build a histogram for the number of ways (or, if you prefer, the probability) to roll each possible sum with four dice, from 4 to 24.

I said *number cubes* consarnit! Get a calculator screen (HOME icon), define $p(x)$ to be some specific interesting polynomial ("define $p(x) = \cdots$"), then tell the calculator:

expand$(p(x)^4)$

Neat Stuff

7. Define $c(n) = \frac{\phi(n)}{n}$.

(a) Make a table for c from 1 to 36, using exact fractional answers.

(b) Complete this table for $c(n)$ looking at powers of primes.

n	$c(n)$
1	
p	
p^2	
p^3	
p^4	

(c) Do you think it is possible for $c(n)$ to be less than 0.1? Explain.

8. Let $d(n) = a(n) \cdot c(n)$, where $a(n)$ is the sum of the reciprocals of the factors of n (see Day 2) and $c(n)$ is defined in Problem 7.

(a) Compute $d(n)$ using any method from 1 to 36, giving decimal answers to four places.

(b) Does d seem to be multiplicative?

(c) Does d seem to have a maximum value? A minimum?

We named this function after David. Or, maybe it was just the next available letter in the alphabet, we forget which...

9. I'd like to solve these, Pat.

Is there an x in the puzzle?

(a) $x^2 + x = 2$ in mod 3

(b) $x^2 + x = 2$ in mod 5

(c) $x^2 + x = 2$ in mod 7

(d) $x^2 + x = 2$ in mod 15

(e) $x^2 + x = 2$ in mod 21

(f) $x^2 + x = 2$ in mod 35

(g) $x^2 + x = 2$ in mod 105

10. (a) If x is 1 mod 3, 0 mod 5, and 0 mod 11, what is it in mod 165?

(b) If y is 0 mod 3, 1 mod 5, and 0 mod 11, what is it in mod 165?

(c) If z is 0 mod 3, 0 mod 5, and 1 mod 11, what is it in mod 165?

(d) Compute $2x + 4y + 8z$, answering in mod 165.

Seven's a lucky number, except perhaps for Brad Pitt and Gwyneth Paltrow.

11. If x is 2 mod 3, 4 mod 5, and 8 mod 11, what is it in mod 165?

12. How many solutions are there to $x^2 = 1$ in mod 165?

13. Which of the following functions are multiplicative?
(a) $b(n) = \frac{1}{n}$
(b) $\sigma_2(n)$, the sum of the squares of the divisors of n
(c) $\chi(n) = \begin{cases} 1 & \text{if } n = 4k+1 \text{ for some positive integer } k \\ -1 & \text{if } n = 4k-1 \text{ for some positive integer } k \\ 0 & \text{if } n \text{ is even} \end{cases}$

That χ function (pronounced "kai" like the Cobra dojo) is pretty wacky. A little *too* wacky if you ask me.

14. Katya says that you can identify a multiplicative function just by declaring what it does to powers of primes. For each description, give a simple rule for the function f.
(a) $f(p^k) = k + 1$
(b) $f(p^k) = 1$
(c) $f(p^k) = p^k$
(d) $f(p^k) = 1 + p + p^2 + \cdots + p^k$

15. Complete this table for $\phi(n)$ and its child.

Remember, the *child* of a function is found by adding its divisors' outputs. For example, $s(27) = \phi(1) + \phi(3) + \phi(9) + \phi(27)$. Sweet child o' ϕ!

n	$\phi(n)$	$s(n)$
1	1	1
p		
p^2		
p^3		
p^4		

16. Sketch a proof that ϕ is a multiplicative function. Your work in today's Important Stuff may be helpful.

17. Prove that if f and g are multiplicative functions, then $h = fg$, the product, is also multiplicative.

18. Last Friday, we saw that the child of the $\phi(n)$ function is the identity function. For example, we saw that

$$\phi(1) + \phi(3) + \phi(5) + \phi(15) = 15$$

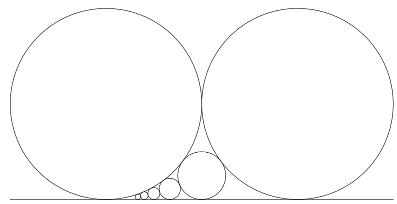
$$\phi(1) + \phi(2) + \phi(7) + \phi(14) = 14$$

Prove that in general,

$$\sum_{d|n} \phi(d) = n$$

Holy notation, Batman! $d|n$ means "d divides n", or that d is a factor of n. This allows the summing to happen only for specific values of d instead of the usual 1, 2, 3, 4...

19. Two circles with diameter 1 are tangent to the x-axis, and mutually tangent at the point $(\frac{1}{2}, \frac{1}{2})$. A big pile of smaller circles are packed in as shown below so they are tangent to the axis and the other circles. Use your mad geometry skills to determine the diameters and the coordinates of the centers of each of these smaller circles.



20. Let $a(n)$ be the sum of the reciprocals of the factors of n. Find a simple rule for the *parent* of this function.

How is a defined? Does this help you find the parent more easily?

21. Yesterday, we said $m(n) = 1$ had a fabulous family, but focused on its children.
(a) Find the parent of m; that is, a function d so that m is the child of d.
(b) Find z, the parent of d.
(c) Describe a general method to find the parent of any multiplicative function.

22. Find the smallest value of $k > 0$ so that each equation has the *maximum possible* number of solutions.
(a) $x^k = 1$ in mod 5
(b) $x^k = 1$ in mod 15
(c) $x^k = 1$ in mod 105

Tough Stuff

23. Find the minimum possible value of $d(n)$, where d is given in Problem 8.

24. Find the maximum possible value of $b(n)$, the sum of the reciprocals of the squares of the divisors of n.

25. Prove that a function is multiplicative if and only if its child is multiplicative.

26. Can a nonzero function be its own ancestor, allowing for more than one generation?

27. Find an identity that shows that if $m = a^2 + b^2 + c^2 + d^2$ and $n = e^2 + f^2 + g^2 + h^2$, then mn can also be written as the sum of four squares.

Clearly, the solution to this problem relies on a deep understanding of the multiplication of quaternions. Um, yeah.

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