Darryl's favorite H*R character is the Poopsmith (no surprise). What's yours?

9 *Circular Reasoning*

Important Stuff

Today's alternate "child" title (child-parent stuff will return soon): Child of Darkness, Child of Light. Does anyone remember that movie? Nope.

PROBLEM

The table below contains the numbers from 0 to 69 arranged so that their remainders after dividing by 7 and 10 are shown in the first column or last row. The numbers from 0 to 10 have already been filled in for you. Fill in the rest of the numbers.

Here's today's Super Password chain: Simpson, food, shape, zodiac, Ford. Got it?

number mod 7										
6							6			
5						5				
4					4					
3	10			3						
2			2							9
1		1							8	
0	0							7		
	0	1	2	3	4	5	6	7	8	9

number mod 10

As an example in the grid, the number 10 is 0 mod 10, and it's 3 mod 7, so it shows up with "coordinates" (0,3). Where should 11 be placed next?

1. (a) Find all solutions to $x^2 - x - 2 = 0$ in mod 7.
   (b) Find all solutions to $x^2 - x - 2 = 0$ in mod 10.
   (c) Use the table in the box above and your answers in mod 7 and mod 10 to quickly find all eight solutions to $x^2 - x - 2 = 0$ in mod 70. Neat, huh?

2. Try making a chart like the one above, using mod 6 and mod 10 instead. What happens when you try to fill in the chart with numbers from 0 to 59?

3. See today's circletastic handout. A path has been drawn from the edge of the largest circle on the right all the way through all the tiny circles. And we mean *all* of them, even

It's circlerific! Today's jokes are not very circlever.

though they keep going like a battery-powered rabbit. The path is straight between centers of consecutive circles.

(a) How long is the piece of the path that goes through the first small circle (in the center of the diagram)?

(b) How long is the piece of the path that goes through the second small circle?

(c) Write an expression (perhaps one with "$\cdots$") that gives the total length of the path, including the diameter running through the largest circle on the right.

Who would win in a hiking contest between Andrew and that bunny? I vote for Andrew.

One with... what? Is this a Buddhism reference?

4. A curved path goes from point A to point B, then point C, then point D. Timon decides to walk in a straight line from A to B, then B to C, then C to D. Is Timon's walk longer or shorter than the path? Always? Explain in brief.

It must be a Buddhism reference, if now we're talking about the curved and straight paths.

5. Function $b(n)$ gives the sum of the reciprocals of the *squares* of the divisors of n. Boy, what a mouthful—an example is far better to see:

$$b(15) = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{225}$$

The divisors of 15 are 1, 3, 5, and 15; so the denominators are $1^2, 3^2, 5^2$, and 15^2.

For each of these, write the entire list of fractions that make up $b(n)$ or the first k terms as listed.

(a) All terms of $b(10)$

(b) All terms of $b(6)$

(c) All terms of $b(24)$

(d) The first 6 terms of $b(60)$

(e) The first 10 terms of $b(2520)$

2520 is a weird choice. What's up with that?

6. (a) Complete this table, where $f(n)$ is the number of solutions to $x^2 = n$ over the integers. Yes, there really are a lot of zeros in this table.

$f(9) = 2$ because $9 = 3^2$ and $9 = (-3)^2$. We count both 3 and -3. Yay! No mods in this problem!

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1		0							2			

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$f(n)$													

(b) Is f multiplicative? Explain.

(c) Like the "dice polynomial" $x + x^2 + x^3 + x^4 + x^5 + x^6$, write a polynomial that expresses the number of different ways to write numbers as a perfect square. (This polynomial will include the term $2x^9 \dots$)

For the purpose of this problem, stop at $2x^{25}$, but keep in mind that this polynomial actually goes on forever with higher-degree terms. This didn't happen with the dice polynomial, which stops at the x^6 term.

7. (a) Use an Nspire or any other method to square the polynomial you found in Problem 6c. Complete this table where $f_2(n)$ is the coefficient of x^n in that squared polynomial.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$f_2(n)$	1		4							4			

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$f_2(n)$	8												

(b) Is f_2 multiplicative? Explain.

8. (a) Find the number of different ways to write 2 as the sum of two squares. Two such ways are $1^2 + (-1)^2$ and $(-1)^2 + 1^2$, but there are two more.

(b) Find the number of different ways to write 7 as the sum of two squares. That was quick!

(c) How many different ways can you write 9?

(d) 13?

(e) 25?

Yes, we mean as the sum of two squares! Writing 9 as $6+3$ or as $\frac{18}{2}$ doesn't count.

Neat Stuff

9. (a) Use the chart from today's box problem to find a, a number that is 1 in mod 10 and 0 in mod 7.

(b) ...and find b, which is 0 in mod 10 and 1 in mod 7.

(c) Calculate $4a + 6b$ in mod 70.

(d) Calculate $7a + 2b$ in mod 70.

10. Here's an interesting circle packing situation. Start with the right-side circle with diameter 1 (labeled "1" in the diagram on the next page) and the circle of diameter $\frac{1}{4}$ from the usual diagram we've been using (labeled "2"). Now, stuff a tangent circle between those, to the right of the circle with diameter $\frac{1}{4}$. Remember, the formula to find the diameter of new circles is

$$2(a^2 + b^2 + c^2) = (a + b + c)^2$$

For a little light reading, we recommend http://mathforum.org/pcmi/hstp/resources/circlepacking. Thanks, Celeste!

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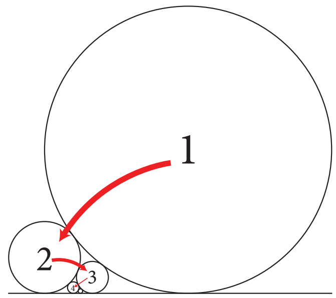
In this formula, a, b, c are the reciprocals of the circles' diameters.



(a) What is this small circle's diameter? (It is labeled "3" above.)

(b) Now stuff another circle to the *left* of that (a tiny "4"). What is its diameter? Note: It's not $\frac{1}{16}$!

(c) Now stuff another circle to the *right* of that (uber-teeny "5"). What is its diameter?

(d) Now stuff another circle to the *left* of that (mega-teensy "6"). What is its diameter?

(e) Notice anything?

11. Prove that if f and g are multiplicative functions, then $h = fg$, the product, is also multiplicative. This means that you must prove that $h(ab) = h(a)h(b)$ given that a and b are relatively prime, and you already know that $h(x) = f(x)g(x)$ for any choice of x, and that f and g are multiplicative.

So $h(ab)$ equals what to start off? Blow it to bits!

12. Complete this table for $c(n) = \frac{\phi(n)}{n}$ and $a(n) = \frac{\sigma(n)}{n}$ along with $d(n) = c(n) \cdot a(n)$.

n	$c(n)$	$a(n)$	$d(n) = c(n)a(n)$
1	1	1	1
p	$1 - \frac{1}{p}$	$1 + \frac{1}{p}$	
p^2			
p^3			

13. Use what you know about the a function to show that for any $k > 0$, no matter how small, there is a value of n for which $c(n) < k$.

14. By hand, find all eight solutions to $x^2 = 3$ in mod 3289.

Okay, we'll be friendly and say that one of the factors of 3289 can be found on any Dr Pepper can.

15. Give an argument that the total area of all the small circles from these circular diagram thingies is finite.

"Thingies" is the technical term for these whozits and whazzamabobs.

16. The centers of the circles from Problem 10 are tending toward a specific point. Find its coordinates.

17. Find the number of different ways to write 25 as the sum of four squares. Two such ways are $0^2 + 0^2 + 5^2 + 0^2$ and $3^2 + 0^2 + (-4)^2 + 0^2$.

Please, please do not write a computer program to do this – think! This question can be answered cleverly using other information from today.

Tough Stuff

18. Let $p(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$, like the dice polynomial, but it goes on forever.

(a) Let $p_2(x) = p(x)^2$, $p_3(x) = p(x)^3$, etc. Write the first few terms of $p_2(x)$, $p_3(x)$, and so forth, until you can describe to someone else what is happening. Then, instead of describing it, help them figure it out.

(b) Let $q_2(x) = \frac{x^2}{(1-x)^2}$, $q_3(x) = \frac{x^3}{(1-x)^3}$, etc. Find the 10th-degree Taylor polynomial expansion of $q_2(x)$, $q_3(x)$, and so forth, about $x = 0$. What is going on here??

19. We now have a method for generating the child of a function; describe a method for generating the parent.

20. Find a formula for the number of ways to write a number as the sum of two squares (including zero and negatives).

21. The triangular numbers are 0, 1, 3, 6, 10, 15, ... Can a number be written as the sum of two triangular numbers in more than one way? Find a formula for the number of ways to write a number as the sum of two triangular numbers.

22. Determine the end behavior of $d(n) = c(n)a(n)$. What are the maximum and minimum possible values for $d(n)$, and when do they occur? (As before, please try to think about this problem analytically—it can and was done without a calculator or computer!)

23. If $m = a^2 + b^2 + c^2 + d^2$, write m^2 as the sum of four squares.

Quat you talking bout, Willis?

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