Welcome!

This packet contains a copy of the problem, the “answer check,” some teaching suggestions, and samples of the student work we received when we first published this PoW. The text of the problem is included below. A print-friendly version is available using the “Print” link on the problem page.

Check out the Problems of the Week blog, visit us on Facebook, and/or follow us on Twitter. You can always find the latest scoop at http://mathforum.org/community/.

Standards

In Cooling Colas, students predict how long it will take cola to cool down to the desired temperature.

If your state has adopted the Common Core State Standards, these alignments might be helpful.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Attend to precision.
6. Use appropriate tools strategically.
7. Look for and make use of structure.

High School: Functions: Building Functions:

1b. Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

High School: Functions: Linear, Quadratic, and Exponential Models:

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).
4. For exponential models, express as a logarithm the solution to \( ab^c = d \) where \( a, c, \) and \( d \) are numbers and the base \( b \) is 2, 10, or \( e \); evaluate the logarithm using technology.

Additional alignment information can be found through the Write Math with the Math Forum service, where teachers can browse by NCTM and individual state standards, as well as popular textbook chapters, to find related problems.

The Problem

Cooling Colas

A few weekends ago we invited some relatives over and served them lunch. The day before we had bought some cans of soda, but we didn’t have enough room in our refrigerator to cool the colas. Since it was February and we live in Pennsylvania, my husband suggested that we leave the cans outside overnight to cool.

The overnight temperature was to be in the twenties, so I was afraid they might freeze; however, I figured out a way to approximate the temperature of the cola at any given time.

First, I read the thermometer in our apartment. It said 72°F. Next I read our outdoor thermometer, and it read 25°F. I put the cola outside for 30 minutes and then brought one of the cans inside to have a drink. Before I drank the soda, I measured the temperature. The temperature of the soda was 60°F.

Based on this information, how long would it take the cola to cool to 35°F? Assume that the outdoor temperature remains constant during the cooling process.

Note: The temperature of the cola is not decreasing linearly.
Extra: Water freezes at 32ºF. Will cola also freeze at this temperature? Explain.

After students submit their solution, they can choose to “check” their work by looking at the answer we provide. Along with the answer itself (which never explains how to actually get the answer) we provide hints and tips for those whose answer doesn’t agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

The cola will reach 35ºF after about 2 hours and 37 minutes.

If your answer does not match our answer,

• Did you get 157 minutes or 2.6 hours? Those are equivalent to 2 hours and 37 minutes.
• Did you notice the note that the cola’s temperature is not decreasing linearly? That means we can’t solve the problem by calculating degrees cooled per minute.
• If you tried to use the fact that the soda cooled 12 degrees in 30 minutes, and scaled up from there, you’re still using a linear model. That doesn’t match the way soda cools.
• If you’ve studied differential equations, you might use the fact that the rate of change in the temperature of an object is proportional to the difference between its temperature and the temperature of its surroundings.
• Did you realize that the temperature of the soda can never get colder than the air around it?
• Did you try sketching a graph of what the temperature over time should look like? Does your graph go down quickly at first and then level out at 25ºF?
• Graphs that go down quickly at first and then level off can be modeled with exponential functions. Did you try graphing equations of the form \( y = ae^{kt} + c \)?
• Have you heard of Newton’s Law of Cooling? If not, you might read this answer from Dr. Math: [http://mathforum.org/library/drmath/view/62887.html](http://mathforum.org/library/drmath/view/62887.html)

If any of those ideas help you, you might revise your answer, and then leave a comment that tells us what you did. If you’re still stuck, leave a comment that tells us where you think you need help.

If your answer does match ours,

• Did you explain any calculus concepts that you used?
• If you used a formula, did you explain how you knew what to plug in where?
• Did you explain any assumptions that you made?
• Is there a hint you could give a student who was struggling to solve the problem?
• Did you describe any “ah-ha!” moments you had?
• Did you describe the strategy you used?

Revise your work if you have any ideas to add. Otherwise leave us a comment that tells us how you think you did—you might answer one or more of the questions above.

The solutions below represent correct solutions that we received from students when we first used this problem. For each, we’ve provided a short note giving a little more insight into the student’s strategy and what we’re on the lookout for as mentors.

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ANSWER: The cola will reach approximately 2.67 hours (i.e. approximately 160 minutes) after it was set outside.

SOLUTION: According to Newton’s law of cooling, the change in temperature of an object with time is described by the equation:

\[
\frac{dT}{dt} = -k(T - S)
\]

where \( T \) is the temperature of the object, \( S \) is the temperature of its surroundings, \( t \) is time, and \( k \) is the cooling constant for the situation.

Multiplying both sides of Eq.(1) by \( dt/(T-S) \) and then integrating both sides gives:

\[
\int(1/(T-S)dT) = \int(-kt)dt
\]

\[
= -kt
\]

Nate used Newton’s Law of Cooling in its “purest” form, a statement about a proportional rate of change. He had to use his differential equations to find a function of temperature in terms of time, which he did using natural log.

I wonder if Nate used properties of logs to rewrite the subtraction in Eq. 3 as ratios, if he could explain how the proportions in Eq. 3 relate to Newton’s Law.
We are given that in 0.5 hour, in surroundings whose temperature was a constant 25 degrees, the tested can of cola cooled to 60 degrees from an initial temperature of 72 degrees. Substituting 0.5 hr for t and 25 degrees for S in Eq.(2) and evaluating the integral over the interval from T = 72 degrees to T = 60 degrees, we get:

(3) \[ -0.5k = \ln(60 - 25) - \ln(72 - 25) = -0.29 \]

Solving Eq.(3) for k gives us that k = 0.58.

To find the time t at which the temperature of the rest of the cola will reach 35 degrees, we substitute 0.58 for k and 25 degrees for S in Eq.(2) and evaluate the integral over the interval from T=72 to T=35:

(4) \[ -0.58t = \ln(35-25) - \ln(72 - 25) = -1.55 \]

Solving Eq.(4) for t tells us that the cola will reach approximately 2.67 hours (i.e. approximately 160 minutes) after it was set outside.

**BONUS**: Cola will not freeze at the same temperature as water, because, although it is mostly water, there are also dissolved in it various substances which have different freezing points than the freezing point of water, such as carbon dioxide gas and sugar.

It will take the cola approximately 2.6 hours to cool off from room temperature of 72 degrees F to 35 degrees F when stored at an outside temperature of 25 degrees F.

Okay, I looked up Newton’s Law of cooling. I first have to find a constant k that is unique to this cola.

I used the formula, which is comes from Newton’s law:

\[ k = \frac{-1}{t} \ln \left( \frac{T(t) - S}{T(0) - S} \right) \]

\( t \) = elapsed time

\( T(t) \) = temperature of object after elapseed time.

\( T(0) \) = intial temperature of object

\( S \) = temperature of the environment surrounding the object.

\( t = .5 \) hr

\( T(t) = 60 \)

\( T(0) = 72 \)

\( S = 25 \)

\[ k = \frac{-1}{.5} \ln \left( \frac{60-25}{72-25} \right) \] which is approx. 0.5896

Through algebra we can resolve the equation for t

\[ t = \frac{-1}{k} \ln \left( \frac{T(t) - S}{T(0) - S} \right) \]

\[ t = \frac{-1}{0.5896} \times \ln \left( \frac{35-25}{72-25} \right) = \text{aprox. 2.6 hours} \]

I found the above formula at the SOS Mathematics site for Diff Eq. I used the search engine on the Math Forum to link to it.

Coca Cola will freeze if kept at 32 degrees long enough. But the carbonation in it keeps the molecules moving for a bit longer in a slushy state before the can explodes in my freezer. If the cola has gone flat it will freeze at 32 degrees since the carbonation has left it. I am sure that the syrup that makes up Cola is thicker that water and the sugar added to it also keeps it from freezing right away.

I think researched Newton’s Law of Cooling, and once he found the formula was able to work quite comfortable with it.

I wonder how Gordon would explain, in words, what the ratio \( (T(t) - S)/(T(0) - S) \) means.

I wonder if Gordon could make an analogy to the slope function, and explain why the natural log of that ratio might be constant in an exponential function, similar to how the slope is constant in a linear function. Perhaps trying some different values for \( T \) and calculating \( (T(t) - S)/(T(0) - S) \) and \( \ln(T(t) - S)/(T(0) - S) \) would help him get started.
It will take 157.486 minutes for the cola to cool to 35 degrees Fahrenheit.

In order to find out how long it will take the cola to cool to 35 degrees I used the equation derived from Newton's Law of Cooling, which says: 

\[ T - T(s) = (T(o) - T(s))e^{-kt}. \]

From the information given in the problem I took 25 degrees F. as \( T(s) \) because that is the surrounding temperature. And I took 72 degrees as \( T(o) \) because that was the temperature in which the cola was originally in. So my equation looked like:

\[ T - 25 = (72 - 25)e^{-kt}. \]

I manipulated the equation so as to isolate \( T \):

\[ T = 25 + 47e^{-kt}. \]

From the information given in the problem I set \( T \) (temperature) as 60 (degrees) and \( t \) (time) as 30. My new equation looked like this:

\[ 60 = 25 + 47e^{-30k}. \]

In order to isolate the \( k \) I had to manipulate the equation by subtracting 25 from both sides:

\[ 35 = 47e^{-30k}, \]

then I divided 47 from both sides:

\[ (35/47) = e^{-30k}, \]

then I isolated the \( -30k \):

\[ \ln(35/47) = -30k, \]

then I isolated the \( k \) by dividing -30 on both sides:

\[ (-1/30)\ln(35/47) \text{ equaled } k. \]

Now that I know what \( k \) equaled I set up a new equation with 35 degrees as my new \( T \) (temperature) and \( t \) (time) as my unknown variable:

\[ 35 = 25 + 47e^{-(-1/30)\ln(35/47)t}. \]

I subtracted 25 on both sides:

\[ 10 = 47e^{-(-1/30)\ln(35/47)t}, \]

then I divided 47 on both sides:

\[ (10/47) = e^{-(-1/30)\ln(35/47)t}, \]

then I isolated the \( t \) by:

\[ \ln(10/47) = -(1/30)\ln(35/47)t, \]

then I divided -((1/30)\ln(35/47)) on both sides to isolate the \( t \):

and with: \((\ln(10/47)) / -((1/30)\ln(35/47)) = t \) I got the answer 157.486254 as my answer.

BONUS: Coke will freeze at a lower temperature. See below for explanation.

Newton's Law of Cooling states that the rate of cooling is directly proportional to the difference in temperature between the object and the surrounding medium. In mathematical terms:

If \( T(t) = \) temperature at a given time \( t \) (minutes)

\[ \frac{dT}{dt} = k(T - 25) \]

\[ 1/(T - 25) \frac{dT}{dt} = k dt \]
integrate:

\[
\int \frac{1}{T - 25} \, dT = \int k \, dt
\]

\[
\ln (T - 25) = kt + C
\]

\[
T - 25 = e^{(kt + C)} = e^{kt} \cdot e^{C}
\]

To calculate \( e^{C} \), I set \( t = 0 \), and \( T(0) \) would be 72:

\[
72 - 25 = 47 = e^{0} \cdot e^{C} = e^{C}
\]

\[
T - 25 = e^{(kt + C)} = 47 \cdot e^{kt}
\]

To calculate \( k \), I set \( t = 30 \), and \( T(30) = 60 \)

\[
60 - 25 = 35 = 47 \cdot e^{(30k)}
\]

\[
35/47 = e^{30k}
\]

\[
30k = \ln 35/47 = -0.2948
\]

\[
k = -0.0098
\]

\[
T - 25 = 47 \cdot e^{(-0.0098t)}
\]

Now I can calculate the time required for the Coke to cool to 35oF:

\[
35 - 25 = 47 \cdot e^{(-0.0098t)}
\]

\[
10/47 = e^{(-0.0098t)}
\]

\[
-0.0098t = \ln 10/47 = -1.5476
\]

\[
t = -1.5476/-0.0098 = 157.486 \text{ min}
\]

BONUS: Coke will freeze at a lower temperature because of the concept of freezing point depression - the more solute particles present in the solvent, the more the f.p. of the solution deviates negatively from the f.p. of the pure solvent. Freezing point depression is a colligative property, meaning that it depends on the number of particles present. This is because the solute particles in the water lower the vapor pressure:

PHASE DIAGRAM FOR WATER

(Y-shaped figure with X, A, T, not to scale)

Under 1 atm of pressure, water freezes at 273K (0oC, 32oF), because that is where the normal freezing curve (a vapor pressure curve, denoted by AX) crosses the 1atm line. However, lowering the vapor pressure (by adding solute) will cause the vapor curve to cross the 1atm line at a lower temperature (lower vapor pressure curve denoted by BY), thus lowering the freezing point.

In the case of Cola, it is a mix of various substances (caffeine, carbon dioxide, carbonic acid, etc.) with molality > 0 (molality = moles solute / kg solvent). The formula for change in f.p. for water is \(-1.86 \text{ K/m} \cdot \text{ molality}\), and since molality is positive, the change in f.p. is negative, making the solution's f.p. less than 32of. (somebody correct me if I'm wrong in my explanation)
It will take approximately 157 minutes for the cola to reach 35F.

**Bonus:** Cola will not freeze at 32F. This is because even though cola has a lot of water in it, it also has other things, such as flavoring, carbon dioxide, phosphoric and citric acids. The presence of dissolved materials, will alter the freezing point and the boiling point of a liquid.

The temperature can be modeled exponentially using the standard formula, \( N(t) = a \cdot e^{kt} \), but a constant must be added because the temperature of the cola can never go below that of the environment, which is 25F. So the formula is now \( N(t) = a \cdot e^{kt} + c \).

- \( N(t) \) is the temperature at time \( t \)
- \( a, k, \) and \( c \) are constants
- \( c \) determines the minimum temperature

We know that the outside temperature is 25F, so we can plug that in for \( c \):

\[
N(t) = a \cdot e^{kt} + 25
\]

At \( t = 0 \), the cola is at 72F because that is the room temperature:

\[
72 = a \cdot e^{k \cdot 0} + 25
72 = a + 25
a = 47
\]

So now the equation becomes:

\[
N(t) = 47 \cdot e^{kt} + 25
\]

Now to find \( k \). We know that at \( t = 30 \), the temperature is 60F, so:

\[
60 = 47 \cdot e^{k \cdot 30} + 25
35 = 47 \cdot e^{k \cdot 30}
35/47 = e^{k \cdot 30}
\ln(35/47) = k \cdot 30
k = \frac{1}{30} \ln(35/47)
\]

The final equation is:

\[
N(t) = 47 \cdot e^{\left(\frac{1}{30} \cdot \ln(35/47)\right) \cdot t} + 25
\]

We have to find \( t \) such that \( N(t) = 35 \), so:

\[
35 = 47 \cdot e^{\left(\frac{1}{30} \cdot \ln(35/47)\right) \cdot t} + 25
10 = 47 \cdot e^{\left(\frac{1}{30} \cdot \ln(35/47)\right) \cdot t}
\ln(10/47) = \left(\frac{1}{30} \cdot \ln(35/47)\right) \cdot t
30 \cdot \ln(10/47) = \ln(35/47) \cdot t
\]

\[
t = \frac{30 \cdot \ln(10/47)}{\ln(35/47)}
\]

Entering this into my graphing calculator, I get: \( t = 157.486 \)

**Mickey**

**age 12**

**Strategy:**

It took 157.486 minutes for the can to cool to 35 F. Bonus: It would not freeze at 32 degrees F because when you dissolve something in water it lowers the freezing temperature as for example in seawater.

Assume that the can cools following Newton’s law of cooling.

**Rate of cooling = constant times difference in temperature between can and outside air**

Suppose the can cools by “delta-T” in a small amount of time “delta-t” and let the constant outside temperature be “To” then

\[
\text{Delta-T} / \text{delta-t} = K \text{(T - To)}
\]

Using Excel I set up a table for the variation of temperature with time for an assumed value of the constant \( K \).

\[
\text{New temperature} = \text{old temperature} + \text{delta-T} = \text{old temperature} + K(T - To) \cdot \text{delta-t}
\]

**Xi did not rely on Newton’s Law. Instead, she realized that the function was exponential, and so she used a general model for exponential functions. She realized that the outside air temperature limits the range of the function, and was able to adjust her model accordingly.**

I wonder how Xi thought to use an exponential function. Did she think about what the temperature graph would look like? How would she explain to a student who was stuck, which model to choose?
At first I made delta-t equal to 1 minute and experimented with K until the temperature at 30 minutes was 60 degrees. Then I changed the increment of time delta-t to 0.1 minutes. I then had to adjust K to again make T(30 minutes) = 60°F. Finally I changed delta-t to 0.01 minutes and K needed to be changed only a slight amount to get the temperature correct at 30 minutes. I then extended my table to find the time when the temperature of the can was 35 degrees F. the attached figure comes from the Excel table and shows the change of temperature with time. The can cooled to 35°F in 157.48

While this problem is classic for any student of calculus, it can be approached from many different angles. Students who have studied exponential functions can use their knowledge of the situation to match a curve to the constraints in the problem. Students who have studied Newton’s Law of Cooling can use one of the many versions of the formula to solve the problem. Students of differential equations can use the fact that the rate of change in the temperature of an object is proportional to the difference between its temperature and the temperature of its surroundings to model the derivative of the time-vs-temperature function, and then use their knowledge of integration or differential equations to figure out what anti-derivative function must be.

As we mentioned in the Common Core State Standard above, one of the factors students must take into account is the effect of the outside temperature. This is not simply a situation where $\frac{dy}{dt} = ry$, or where $y = ae^{rx}$. It is possible to fit a curve of the form $y = ae^{rx}$ to the data given. The graph looks something like

It’s clear that if we use the $y = ae^{rx}$ model, it shows the cola getting colder than the air around it, a physical impossibility. One way to shift the horizontal asymptote up is to add a constant, and in fact, $y = ae^{rx} + c$ models the situation nicely (specifically, $y = 47e^{0.0098x} + 25$).

In dealing with the differential equation, because the rate of change of the temperature is not proportional to the temperature but rather to the difference between the temperature and the outside
air, it would seem that we’d need to work with the differential equation \( \frac{dy}{dt} = k(y - 25) \) with \( y(t) \) being the temperature at a given time and 25 being the temperature of the outside air. I wasn’t sure how to find the anti-derivative of that directly, although several of the students above managed it nicely! The good news for me was, I could use the handy trick mathematicians use when they Solve a Simpler Problem and Look for and make use of structure (as it says in the Common Core Standards for Mathematical Practice). Rather than focusing on finding the function \( y(t) \) directly, I focused on finding a function that models the difference between the outside air temperature and the soda, because that’s the function that’s directly proportional to the rate of change. If we call that function \( d(t) \), we know \( d(t) = y(t) - 25 \), or \( y(t) = d(t) + 25 \). We know \( \frac{dd}{dt} = kd \), which suggests that \( d(t) = d_0 e^{kt} \). From there, it’s not too hard to find \( d_0 \), which is just the difference between the soda’s initial temperature and the outside temperature, nor is it too hard to use \( d(t) \) to find \( y(t) \), because we can simply add 25 to \( d(t) \).

Depending on your students’ prior knowledge, what source they find, or what you’ve taught them, they may have different formulas associated with Newton’s Law of Cooling. The student solutions presented above demonstrate a wide range of formats for Newton’s Law of Cooling, from Gordon’s that was already solved for \( k \), to Sheila’s that was already in exponential form, to Simon’s that was a differential equation. Or, your students may not know of Newton’s Law of Cooling, but they will be able to use their concept of exponential decay to model the situation with a different representation, such as a general exponential function (like Xi) or a table and graph (like Mickey). Mickey’s solution in particular shows how conceptual understanding and good problem-solving strategies and skills (like Make a Table and Change the Representation) can allow you to solve a problem, even when you don’t know any of the common procedures taught in Calculus class.

The Online Resources Page for this problem contains links to related problems in the Problem Library and to other web-based resources.

If you would like a calendar of the Current Problems, consider bookmarking this page:

http://mathforum.org/pow/support/

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A Trig & Calc PoW-specific rubric can be found linked from the problem to help in assessing student solutions. We consider each category separately when evaluating the students’ work, thereby providing more focused information regarding the strengths and weaknesses in the work. We strove to make it generic and student-friendly. We encourage you to share it with your students to help them understand our criteria for good problem solving and communication.

We hope these packets are useful in helping you make the most of Trig & Calculus Problems of the Week from the Library. Please let me know if you have ideas for making them more useful.

~ Max and Brianna <max@mathforum.org>