The Math Forum: Problems of the Week

Problem Solving and Communication Activity Series

Round 18: Change the Representation

All math problems, whether they are word problems, arithmetic problems, equations to solve, etc., come to us in a particular representation. Word problems are represented in story form, using words and often referencing a particular context. Arithmetic problems are represented numerically. Equations are represented using mathematical symbols. Each representation has benefits to the problem solver. For example, word problems allow students to apply their knowledge of the given context, which can allow them to check that their approaches are reasonable. Numeric and symbolic representations can make it easy for students to manipulate objects in the problem, and to quickly see patterns. Visual and physical representations, such as manipulatives, diagrams, and graphs, can often help students gain new insights into the problem and provide them with additional tools for solving it. Changing the representation can mean use of a different form of representation (e.g. using a line drawing for a word problem) or it can mean trying different ways of presenting the information in the same form (e.g. rewriting all of the numbers as fractions with a numerator of 1). Considering multiple representations and choosing representations that fit the problem well are important problem-solving skills.

The activities below help students to brainstorm and work with multiple representations, and compare what they learned about the problem using their different representations.

The activities are written so that you can use them with problems of your choosing. We include a separate section afterward to show what it might look like when students apply these activities to the current Math Fundamentals Problem of the Week.

Problem-Solving Goals

Changing the representation of a problem can help problem-solvers:

- Strengthen their understanding of the problem.
- Gain new insights into the problem or solution.
- Provide additional tools for solving the problem.
- Find multiple solution paths, leading to deeper mathematical connections.

Communication Goals

Changing the representation of a problem requires that students change how they communicate about the problem and find different ways to express the same idea or information. They might:

- Paraphrase the problem in terms of a different representation.
- Re-tell the story of the problem with a different context.
- Organize the numerical and calculation strategies using a table or other organizational method.
- Use mathematical symbols to restate the problem succinctly.
- Use diagrams to communicate the math in the problem.
- Represent the problem graphically.

Activities

I. Brainstorm

Format: students working individually or in pairs, then sharing with groups of 4-6.

There are many ways to represent math problems and mathematical ideas. Math problems are often represented in words. Math can be represented visually, through graphs, diagrams, and sketches. Tables and expressions can be used to represent math ideas numerically. Mathematical ideas are often represented symbolically, with operations, numbers, variables, and functions. Each representation can help you understand and solve the problem in different ways. Problems represented in words help you to make sense of the problem and use your knowledge of real-world situations. Graphical representations can lead to new insights or problem-solving methods. Numerical representations can help you find patterns and generate strategies. Symbolic representations represent math ideas clearly and succinctly, and help you to manipulate the mathematical objects.

Sample Activity

Work individually or in pairs to begin filling in the blanks of the following prompts for just a few minutes. Then share ideas with the larger group of 4-6 students. The first question asks you to think about the math ideas in the problem, which might get you thinking about representations you remember. The second asks you to think as creatively as you can.

1) The main mathematical ideas and relationships in this problem are _______________________________.

2) I could represent this problem by: (drawing a picture, using some blocks, making a graph, writing some equations, telling a different story, writing the numbers or expressions in a different way).

3) My representation might look like ___________________________________.

Share your thoughts with your group. If the ideas you hear spark other ideas, record those too. Try to brainstorm as many possible representations as you can.

Suggestions

These suggestions can be used as a place to start with a student who is struggling.

- Think of or make up another story or situation that could be used to present a math problem just like this.
- Think of manipulatives like blocks, chips, *algeblocks*, fraction bars, the number line, or a drawing that you could use to model and explore this problem.
- Think about substitutions you could make for some of the quantities in the problem (rewriting whole numbers as fractions; express all of the quantities in terms of the smallest item; use a different expression that has the same value but might make the calculations easier).

Key Outcomes

- Identify key mathematical ideas in the problem that can be represented.
- Think of ways that mathematical ideas are sometimes represented.
- Generate multiple possible representations for the problem to be solved.

II. Representing

Format: Individually or in pairs.

The focus in this activity is on using writing to organize the problem solving activity, to notice patterns or ideas that make the solution possible, and to ask specific questions that need to be answered in order to make progress,

Sample Activity

Step 1: Pick the representation that seems most useful or stimulating and play it out in detail. Use your new representation to explore the problem and solve it if you can.

Step 2: Explain to your partner how your new representation works:

o What did you notice about the problem when you changed the representation? What new information, relationships, patterns, and approaches occurred to you?

o What have you been able to figure out for a solution so far?

o If you are stuck, which representation shows best where and why you are stuck? Ask your partner(s) if they have a way of representing the problem that helps you get unstuck.

Step 3: Check your solution with the original problem to make sure it fits with all of the information and constraints of that situation..

Key Outcomes

- Play out a particular representation as fully as possible.
- Consider multiple representations when you get stuck.
- Work with others to see multiple perspectives and fresh ideas.

III. Comparing

Format: Students working individually or pairs and then sharing with the whole group.

Sample Activity:

Step 1: Within your group of 4-6, compare the different solution paths you generated. Did different representations lead to different insights? In what ways are the different solutions similar?

Step 2: If you can, pick a representation or solution path that you think best fits the problem to write up to share with the class. If multiple representations fit the problem as well, write up how each representation helped you get to a key insight into the problem.

Step 3 (optional): Submit your write-up to the PoW online.

Step 4 (optional): Use a jigsaw or gallery walk format to share your explanation with classmates and to appreciate their insights.

Key Outcomes:

- Compare insights generated by multiple representations. Identify the similarities and the differences in the contributions from each representation.
- Evaluate how well different representations fit a given problem. Figure out how to recognize when a particular way of changing the presentation of the problem will be useful.

Examples: Sports Weigh In (FunPoW)

The goal of these lessons is for the students to reflect on their own process in developing simpler versions of a problem. While it's tempting to steer them towards certain key ideas, we want students to experience the gain in confidence that comes from being able to rely on their own resources in order to get going. As a result, we tend to hold back on suggestions and focus on supporting the student's own thinking. If students are stuck, or we feel their ideas need further probing and clarifying, we might help with facilitating questions that reinforce the problem-solving strategies. Check out the "funpow-teachers" discussion group (http://mathforum.org/kb/forum.jspa?forumID=526) for conversations about this problem in which teachers can share questions, student solutions, and implementation ideas.

If we do facilitate by asking some strategy questions, then at the end of the session we often ask students to notice the questions and suggestions we asked so that they can begin to do that for themselves: Which were helpful? Could you see how you could use these with other problems? Which questions would you put on a class list of "Ways to get Unstuck in Changing the Representation"?

I. Brainstorming

The main mathematical ideas and relationships in this problem are:

- The combination of balls on each scale weighs a specific weight.
- The weight of all the soccer balls is exactly the same.
- The weight of all the baseballs is exactly the same.
- The weight of all the tennis balls is exactly the same.
- The heaviest combination shown is 2 soccer balls plus 2 tennis balls (28 ounces).
- The lightest combination is 1 of each ball (19 ounces).
- Two baseballs plus 1 soccer ball weigh 22 ounces.

I could represent this problem with/by:

- A table of information.
- A diagram for keeping track of the balls and weights.
- Writing the relationships a different way.

My representation might look like:

- Creating a table to keep track of the combinations (I might use a computer spreadsheet to do this).
- Drawing out the scales and the combinations and using my diagram to determine the weight of each ball.
- Replacing the balls on a scale with other types of balls and seeing if I can figure out the new weight. Comparing the weights of different types of balls.

II. Representing

Representation 1

I created a table with five columns. The middle three columns represent the number of balls on each scale, while the fifth column is the total weight for each scale. I used the pictures to write the first three rows, but I put them in order from lightest to heaviest to see if I noticed a pattern.

Once I had the first three rows, I decided to double the first row, since there was one of each ball on that row. That is row number 4 in my table. By making two of each ball, the total weight also doubled.

I realized that I could compare row 3 with row 4. The only difference is the two baseballs. I can see that having 2 baseballs on the scale adds 10 ounces, so 1 baseball is half this amount, or 5 ounces.

Since 2 baseballs weigh 10 ounces, I can now use row 2 and figure out that 1 soccer ball is 12 ounces because I subtracted the 10 ounces (2 baseballs) from the total of 22 ounces.

Then using row 1, if a baseball weighs 5 ounces, and a soccer ball weighs 12 ounces, a tennis ball has to weigh 2 ounces so that together they all weigh 19 ounce.

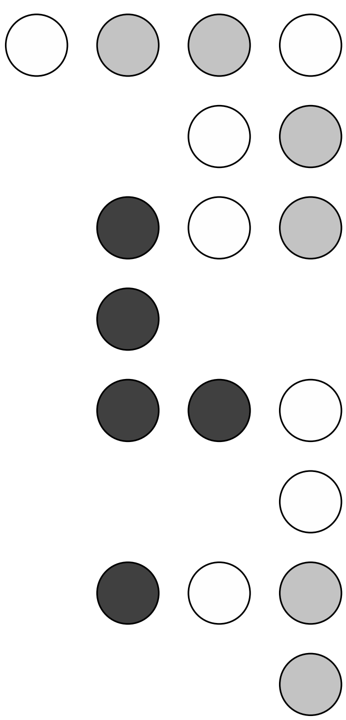
Row	Number of Baseballs	Number of Soccer Balls	Number of Tennis Balls	Total Weight on Scale
1	1	1	1	19 ounces
2	2	1	0	22 ounces
3	0	2	2	28 ounces
4	2	2	2	38 ounces

Representation 2

When I looked at the scales I noticed that because there are two of each type of ball on scale two, maybe I could think about splitting it in half. I decided to use a diagram of the balls on the scale to help me think about it visually.

I used different colors to represent each kind of ball: a soccer ball is white, a tennis ball is gray, and a baseball is black.

I kept drawing diagrams using the information I learned from the previous step. I wrote my notes on the side of my diagrams below so you could follow my work. I found out that a baseball was 5 ounces, a soccer ball was 12 ounces and a tennis ball was 2 ounces.



Equals 28 oz. (scale 2)

Equals 14 oz. because it's half of 28 oz.

Equals 19 oz. (scale 1)

Equals 5 oz. because 19 oz. - 14 oz. = 5 oz.

Equals 22 oz. (scale 3)

Equals 12 oz. because 5 oz. + 5 oz. + 12 oz. = 22 oz.

Equals 19 oz. (scale 1 again)

Equals 2 oz. because 5 oz. + 12 oz. + 2 oz. = 19 oz.

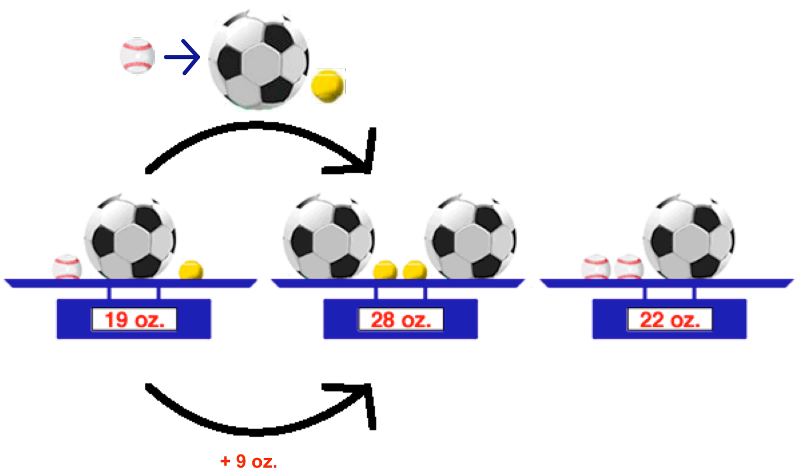
Representation 3

Instead of focusing on the totals on each scale, I decided to focus on each type of ball. I wondered which type weighed more, and what would happen if you swapped out different types of balls.

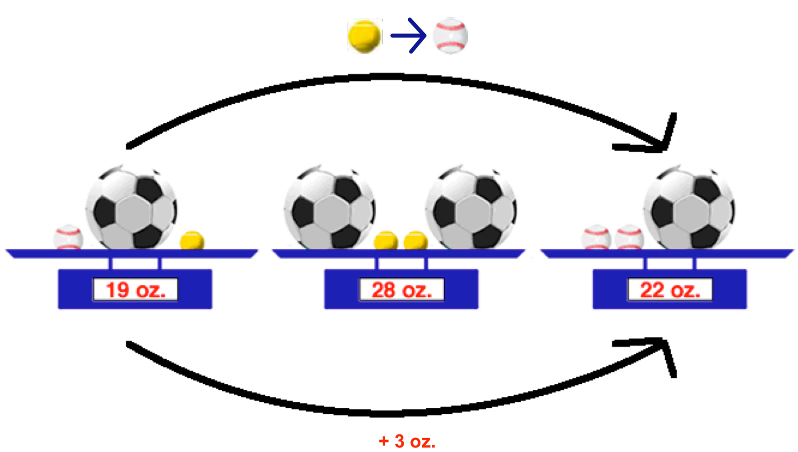
I looked at the three scales:

And decided that:

A soccer ball + tennis ball weigh 9 oz. more than a baseball (comparing scale 1 and scale 2).



A baseball weighs 3 oz. more than a tennis ball (comparing scale 1 and scale 3).



So I can replace all the tennis balls with baseballs as long as I increase the weight by 3 oz. each time.

Now it's easy to see that a soccer ball weighs 34 oz. – 22 oz. (the difference between scale 2 and scale 3). So a soccer ball weighs 12 oz.

If a soccer ball weighs 12 oz, then:

So a baseball weighs 5 oz.

A tennis ball weighs 3 oz. less than a baseball, so a tennis ball weighs 2 oz.

III. Comparing

We think the three representations show three different methods for finding for the weight of each ball but there were some similarities too

Similarities:

In all three representations we went step-by-step, using the information that is found from one calculation or step with the next step. All three methods show how keeping track of the information from each step is important because it makes it easier to compare and to think of other ways to change it.

Differences:

In the first representation, the table helped us see that when we doubled the balls in row 1, then we had 2s in each column for each ball, and this was similar to row 3. We could see that row 3 had no baseballs, but two of everything else. This means that two baseballs were the difference in the two rows. From there we could find how much the baseballs weighed.

In the second representation, the baseballs were still the first ball whose weight we found, but the steps were different. We used scale 2 was used to find out how much a tennis ball plus a soccer ball weighed, and we compared the weight to scale 1 because a baseball plus one of each of the other 2 balls was 19 ounces. This meant the baseball was 5 ounces, and this information was used with scale 3. We compared the scales visually instead of in a table, which is why we thought about splitting instead of doubling.

In the third representation, we still found the weight of the baseballs first, but we found it by getting rid of the tennis balls. We changed what was on each scale instead of doubling or halving an entire scale's load. Changing how the problem was written by thinking about comparing the types of balls helped us to do that.