The Math Forum: Problems of the Week

Problem Solving and Communication Activity Series

Round 18: Change the Representation

All math problems, whether they are word problems, arithmetic problems, equations to solve, etc., come to us in a particular representation. Word problems are represented in story form, using words and often referencing a particular context. Arithmetic problems are represented numerically. Equations are represented using mathematical symbols. Each representation has benefits to the problem solver. For example, word problems allow students to apply their knowledge of the given context, which can allow them to check that their approaches are reasonable. Numeric and symbolic representations can make it easy for students to manipulate objects in the problem, and to quickly see patterns. Visual and physical representations, such as manipulatives, diagrams, and graphs, can often help students gain new insights into the problem and provide them with additional tools for solving it. Changing the representation can mean use of a different form of representation (e.g. using a line drawing for a word problem) or it can mean trying different ways of presenting the information in the same form (e.g. rewriting all of the numbers as fractions with a numerator of 1). Considering multiple representations and choosing representations that fit the problem well are important problem-solving skills.

The activities below help students to brainstorm and work with multiple representations, and compare what they learned about the problem using their different representations.

The activities are written so that you can use them with problems of your choosing. We include a separate section afterward to show what it might look like when students apply these activities to the current Geometry Problem of the Week.

Problem-Solving Goals

Changing the representation of a problem can help problem-solvers:

- Strengthen their understanding of the problem.
- Gain new insights into the problem or solution.
- Provide additional tools for solving the problem.
- Find multiple solution paths, leading to deeper mathematical connections.

Communication Goals

Changing the representation of a problem requires that students change how they communicate about the problem and find different ways to express the same idea or information. They might:

- Paraphrase the problem in terms of a different representation.
- Re-tell the story of the problem with a different context.
- Organize the numerical and calculation strategies using a table or other organizational method.
- Use mathematical symbols to restate the problem succinctly.
- Use diagrams to communicate the math in the problem.
- Represent the problem graphically.

Activities

I. Brainstorm Representations

Format: students working individually or in pairs, then sharing with groups of 4-6.

There are many ways to represent math problems and mathematical ideas. Math problems are often represented in words. Math can be represented visually, through graphs, diagrams, and sketches. Tables and expressions can be used to represent math ideas numerically. Mathematical ideas are often represented symbolically, with operations, numbers, variables, and functions. Each representation can help you understand and solve the problem in different ways. Problems represented in words help you to make sense of the problem and use your knowledge of real-world situations. Graphical representations can lead to new insights or problem-solving methods. Numerical representations can help you find patterns and generate strategies. Symbolic representations represent math ideas clearly and succinctly, and help you to manipulate the mathematical objects.

Sample Activity

Work individually or in pairs to begin filling in the blanks of the following prompts for just a few minutes. Then share ideas with the larger group of 4-6 students. The first question asks you to think about the math ideas in the problem, which might get you thinking about other representations you know. The second asks you to think as creatively as you can.

1) The main mathematical ideas and relationships in this problem are ________________________________.

2) I could represent this problem by: (drawing a picture, using some blocks, making a graph, writing some equations, telling a different story, writing the numbers or expressions in a different way).

3) My representation might look like _____________________________________.

Share your thoughts with your group. If the ideas you hear spark other ideas, record those too. Try to brainstorm as many possible representations as you can.

Suggestions

These suggestions can be used as a place to start with a student who is struggling.

- Think of or make up another story or situation that could be used to present a math problem just like this.
- Think of manipulatives like blocks or chips or algeblocks or fraction bars or the number line or a drawing that you could use to model and explore this problem.
- Think about substitutions you could make for some of the quantities in the problem (rewriting whole numbers as fractions; express all of the quantities in terms of the smallest item; use a different expression that has the same value but might make the calculations easier)

Key Outcomes

- Identify key mathematical ideas in the problem that can be represented.
- Think of ways that mathematical ideas are sometimes represented.
- Generate multiple possible representations for the problem to be solved.

II. Representing

Format: Individually and then in pairs or teams.

The focus in this activity is on using writing to organize the problem solving activity, to notice patterns or ideas that make the solution possible, and to ask specific questions that need to be answered in order to make progress,

Sample Activity

Step 1: Pick the representation that seems most useful or stimulating and play it out in detail. Use your new representation to explore the problem and solve it if you can.

Step 2: Explain to your partner how your new representation works:

- What did you notice about the problem when you changed the representation? What new information, relationships, patterns, and approaches occurred to you?
- What have you been able to figure out for a solution so far?
- If you are stuck, which representation shows best where and why you are stuck? Ask your partner(s) if they have a way of representing the problem that helps you get unstuck.

Step 3: Check your solution with the original problem to make sure it fits with all of the information and constraints of that situation.

Key Outcomes

- Play out a particular representation as fully as possible.
- Consider multiple representations when you get stuck.
- Work with others to see multiple perspectives and fresh ideas.

III. Comparing

Format: Students working individually or pairs and then sharing with the whole group.

Sample Activity:

Step 1: Within your group of 4-6, compare the different solution paths you generated. Did different representations lead to different insights? In what ways are the different solutions similar?

Step 2: Select the representations that provided the most insight into the problem or key steps in the solution. Prepare a presentation for the class on how each representation helped you get to a key insight into the problem.

Step 3 (optional): Submit your write-up to the PoW online.

Step 4 (optional): Use a jigsaw or gallery walk format to share your explanation with classmates and to appreciate their insights.

Key Outcomes:

- Compare insights generated by multiple representations. Identify the similarities and the differences in the contributions from each representation.
- Evaluate how well different representations fit a given problem. Figure out how to recognize when a particular way of changing the presentation of the problem will be useful.

Examples: Flying High (GeoPoW)

The goal of these lessons is for the students to reflect on their own process in developing simpler versions of a problem. While it's tempting to steer them towards certain key ideas, we want students to experience the gain in confidence that comes from being able to rely on their own resources in order to get going. As a result, we tend to hold back on suggestions and focus on supporting the student's own thinking. If students are stuck, or we feel their ideas need further probing and clarifying, we might help with facilitating questions that reinforce the problem-solving strategies. Check out the "geopow-teachers" discussion group (http://mathforum.org/kb/forum.jspa?forumID=529) for conversations about this problem in which teachers can share questions, student solutions, and implementation ideas.

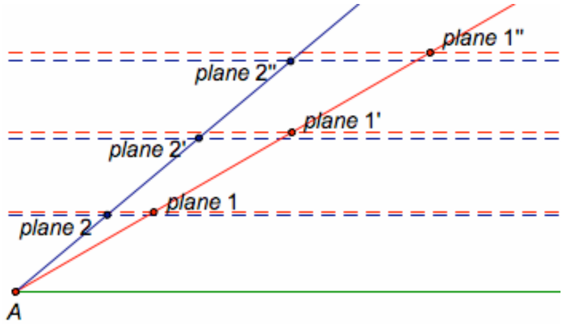
If we do facilitate by asking some strategy questions, then at the end of the session we often ask students to notice the questions and suggestions we asked so that they can begin to do that for themselves: Which were helpful? Could you see how you could use these with other problems? Which questions would you put on a class list of "Ways to get Unstuck in Changing the Representation"?

I. Brainstorm Representations

- The mathematical ideas in this problem are:
  - Rate (mph).
  - Height (altitude).
  - Angle.
  - Slope.
- I could represent this problem by:
  - Drawing a picture.
  - Acting the problem out.
  - Making a graph.
  - Writing equations using variables.
- My representation might look like:
  - Two slanted lines representing the airplanes rising over time.
  - Moving pieces along two different slanted lines or ramps, one moving quicker and one moving slower.
  - A graph of lines with different slopes.
  - Equations representing the height over time of each airplane using the d=rt relationship.

II. Representing

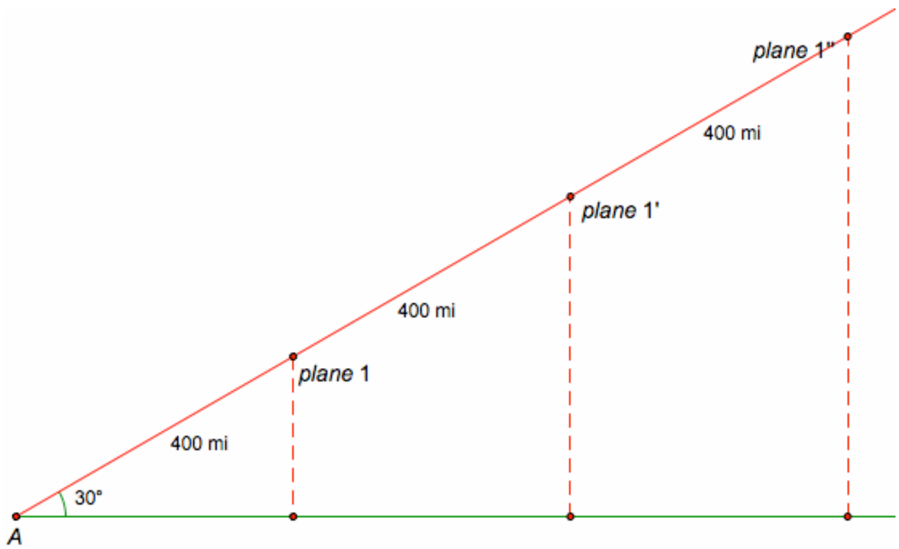
- **Student 1:** I am going to draw a picture and see how that helps me solve the problem.

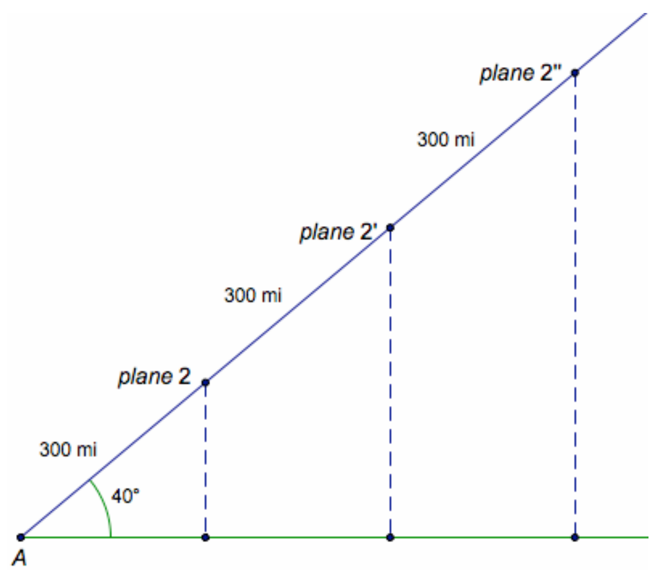


My picture shows the planes rising. Each point represents where the plane is after one hour, two hours, etc.

It is drawn to scale, so I am pretty sure that the plane that is going faster at a smaller angle is actually rising more each hour, but I need to figure out how much more altitude the fast plane gains each hour.

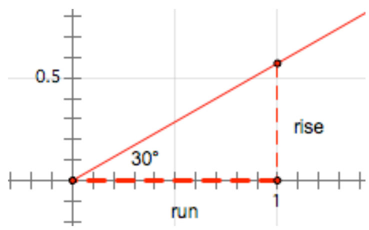
I added the information I used to find to my picture, and labeled stuff. The picture was pretty messy, so I broke it apart and made different pictures for plane 1 and plane 2.

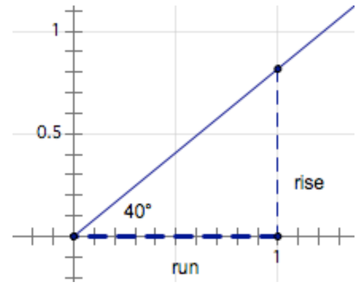




• **Student 2:** I am going to represent the airplane paths as lines with different slopes.

The slope of plane 1's path is made by the 30 degree angle and the slope of plane 2's path is made by the 40 degree angle. I need to figure out how to turn that into the rise over run form of the slope that I know.



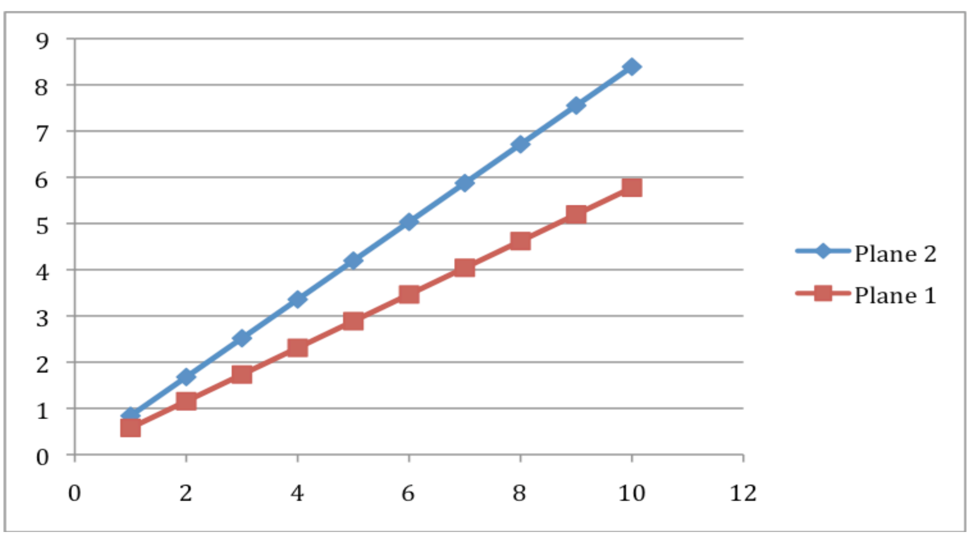


Well I can tell that the 40-degree plane has a bigger rise. I think I can use trigonometry to find what the rise is.

For plane 1 $\tan(30°)=\frac{rise}{run}$, so that means the tangent of 30 degrees is the slope. It must be the same for plane 2, the slope is $\tan(40°)$.

I made a table and plotted points for each plane's line where y is the rise (altitude) I want to compare:

x	Plane 1 y = x*tan(30)	Plane 2 y = x*tan(40)
1	0.577350269	0.839099631
2	1.154700538	1.678199262
3	1.732050808	2.517298894
4	2.309401077	3.356398525
5	2.886751346	4.195498156
6	3.464101615	5.034597787
7	4.041451884	5.873697418
8	4.618802154	6.712797049
9	5.196152423	7.551896681
10	5.773502692	8.390996312



At each x-value, the y-coordinate of plane 2 is higher, so I think it is gaining altitude more quickly.

- **Student 3:** I tried to express the problem with equations, specifically with rates, since the problem is asking "how fast?"

  Quantities:

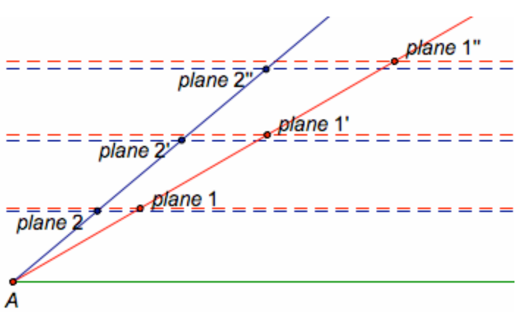
    - Plane 1 angle of ascent (30 degrees).
    - Plane 2 angle of ascent (40 degrees).
    - Plane 1 airspeed (400 mph).
    - Plane 2 airspeed (300 mph).
    - Plane 1 horizontal/ground speed (unknown).
    - Plane 2 horizontal/ground speed (unknown).
    - Plane 1 vertical speed or change in altitude in mph (unknown).
    - Plane 2 vertical speed or change in altitude in mph (unknown).

  Relationships:

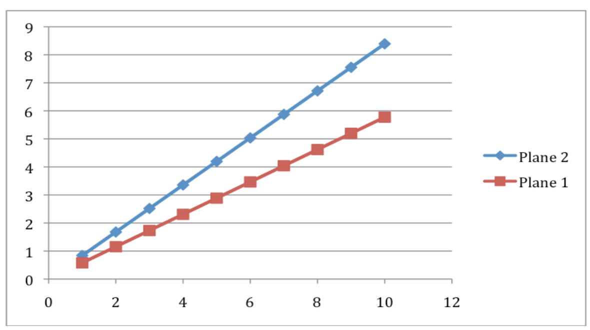
    - Distance = rate * time.
    - Plane 1's angle of ascent is 10 degrees less than plane 2's.
    - Plane 1's speed is 100 mph more than plane 1's.
    - Plane 1's angle is 3/4 of plane 2's angle.
    - Plane 1's speed is 4/3 of plane 2's speed.
    - After 1 hour, plane 1 has gone 400 miles and plane 2 has gone 300 miles. After 2 hours, plane 1 has gone 800 miles and plane 2 has gone 600 miles.
    - After x hours, plane 1 has gone 400x miles and plane 2 has gone 300x miles along its path of ascent.
    - Vertical speed = change in height per hour.
    - Plane 1 rising at an angle of 30 degrees means its height increases less with each mile it travels than Plane 2's height does.
    - But plane 1 traveling more quickly means that it is traveling more miles each hour.
    - I wonder if I need to figure out which has the bigger effect on change in height per hour, the angle or the speed.

III. Comparing

When we compared our solutions we noticed that different representations caused us to get different answers. **Student 1** and **Student 2** got different answers. The first thing we needed to do was to see why. Did one of their representations change the original problem?



Student 1



Student 2

The lines are both the same, rising at the same angles. In Student 1's representation, the planes are at different places on the x-axis at the same time (the points represent the same moment in time). In Student 2's representation, the planes are always at the same place on the x-axis. We aren't sure if the points are at the same moment in time.

Student 1: I think my representation is right because I represented the angles and the miles per hour, which is all of the information given in the problem. The angles are represented by the angles of each line from the ground, and the miles per hour are represented by the fact that plane 2 travels along its line only 3/4 of the distance plane 1 travels every hour.

My picture is like if you took pictures of the planes at different times and lined them up so you could see the path. It's a picture of where the planes actually were at different times.

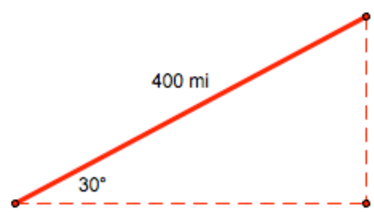
Student 2: My graph is based on the calculation y = x * tan(angle of ascent). If you think about slope as rise over run, y is the rise and x is the run. x and y both have the same units, miles, because they measure distance traveled. So my graph shows the paths of the planes, too. It shows how high the plane is when it's gone a certain distance in the x-direction.

I don't think time is represented in my graph, because everything is in miles, not miles per hour. It just shows that if they travel a certain distance horizontally, what their vertical height is. To take time into account, I would need to figure out at a given time where each plane was on the x-axis, and then see what their height was at that time.

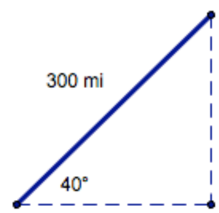
Student 3: Yeah, one thing I noticed was that even though plane 2 rises more with each mile they travel, they are traveling fewer miles each hour. So you need to take both things into account.

Writing up the Combined Solution

Representation 1: Visual (Right Triangle Trigonometry)



Plane 1: Traveling 1 hr at 30° and 400 mph



Plane 2: Traveling 1 hr at 40° at 300 mph

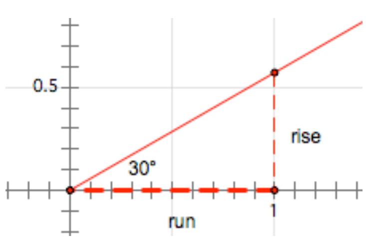
Using this representation we were able to use trigonometry to determine that in one hour plane 1 will travel vertically $400\sin(30°)$ and horizontally $400\cos(30°)$. Plane 2 will travel vertically $300\sin(40°)$ and horizontally $300\cos(40°)$. Distance traveled in one hour is the same as speed in miles per hour, so those values represent the horizontal and vertical speeds of each plane.

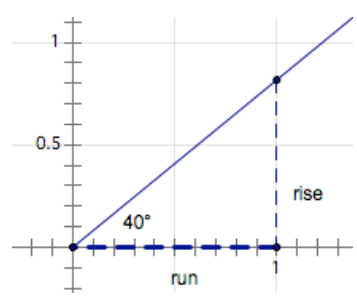
The vertical speed of a plane, in general, is airspeed*sin(angle of ascent). We calculated it out and saw that $400\sin(30°)$ = 200 mph and $300\sin(40°)$ = 193 mph, so plane 1 is traveling 7mph faster, vertically. That means after each hour it has gained 7 miles in height on plane 2, so after two hours it would be 14 miles higher, after 3 hours it would be 21 miles higher, etc. Although in real life, planes don't usually keep gaining altitude at a steady rate for hours, or they'd be in orbit!

Representation 2: Slopes, Equations, and Right Triangle Trigonometry

From Student 1's representation, it's clear that you can calculate vertical speed directly, by figuring out the change in height of each plane per hour (r = d/t).

But we also noticed that there is another way to look at the speed: a combination of the plane's horizontal speed and the amount of height it gains per horizontal mile.





As Student 2 figured out, the $\frac{rise}{run}$ (amount of height it gains per horizontal mile) of each plane is equal to the tangent of its angle of ascent.

Now we have to figure out the plane's groundspeed, or how much it "runs" when it travels for an hour.

For that, we used Student 1's representation, and figured out that for plane 1, the groundspeed is $400\cos(30°)$ mph and for plane 2 the groundspeed is $300\cos(40°)$ mph.

What does that tell us? Well, plane 1 is traveling along the x-axis at $400\cos(30°)$ mph, and for each mile it travels horizontally it gains $\tan(30°)$ miles. So its vertical speed could be found by $400\cos(30°)$ * $\tan(30°)$, which is the same as $400\sin(30°)$.

Plane 2 is traveling along the x-axis at $300\cos(40°)$ mph, and for each mile it travels horizontally it gains $\tan(40°)$ miles. So its vertical speed could be found by $300\cos(40°)$ * $\tan(40°)$, which is the same as $300\sin(40°)$.

We get the same answer, even though we were thinking of speed as being a combination of two factors.

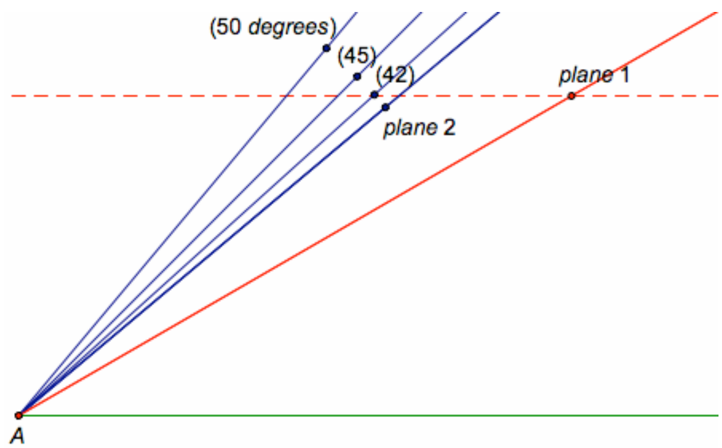
What we Learned from each Representation

Student 1's visual representation was really important. We needed to be able to see the trigonometry relationships to solve the problem, no matter what other representations we used.

Student 2 and Student 3 thinking together about slope and rates helped us understand more of the *why* behind the problem. They helped us see that plane 1 gains less altitude for each mile it travels, but it travels more miles per hour, and plane 2 gains more altitude for each mile it travels, but it travels fewer miles per hour. So it turns out, for some reason, that the change in speed affects the altitude more than the change in slope.

We decided to work together to explore the Extra, and to first try to figure out what happens when we change the angle, and then try to figure out what happens when we change the speed.

Extra (Changing the Angle):



We used drawings to experiment with the angle first. Plane 1 gained altitude more quickly in this problem so we left it where it was and changed the angle of plane 2 until it was at the same height as plane 1. It looked like at about 42 degrees, the angle would be perfect.

We then used algebra to find the angle more exactly by using the expressions for y (the altitude) for each plane:

$$300\sin(x) = 400\sin(30)$$

$$\sin(x) = \frac{4}{3}\sin(30)$$

$$\sin(x) = \frac{4}{6}$$

Using inverse sine, we found out that x = 41.8 degrees.

We noticed the 4/3 in our calculation and remembered that the ratio of the speed of plane 1 to that of plane 2 is 4/3. It makes sense that in order for the vertical speed to be the same, the ratio of the sines should be the inverse, because you multiply airspeed and the sine to get the vertical distance.

Extra: Changing the Speeds

For this problem we said let's see how much slower plane 1 has to go to have the same vertical speed as plane 2. We decided to check what the ratio of the sines of 40 degrees and 30 degrees was, and see if, when the speed ratio was the reciprocal of that, we'd get equal ascents.

$$\frac{\sin(40)}{\sin(30)} = 1.28557522$$

$$\frac{1}{1.28557522} = 0.777861913$$

$$\frac{300}{x} = 0.777861913$$

x = 385.672566

Check:

300*sin(40) = 192.836283

385.672566*sin(30) = 192.836283

Awesome!

Final reflection:

Maybe part of the reason that the speed has a bigger effect than the angle is that something gets done to the angle (you take the sine of it) and maybe that changes the effect of changing the angle.